

Numerical Methods in Scientific Computing



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First edition 2005, improved 2008

Published by VSSD

Leeghwaterstraat 42, 2628 CA Delft, The Netherlands

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internet: <http://www.vssd.nl/hlf>

URL about this book: [**http://www.vssd.nl/hlf/a002.htm**](http://www.vssd.nl/hlf/a002.htm)

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Printed version

ISBN-13 978-90-71301-50-6

Electronic version

ISBN-13 978-90-6562-179-5

NUR 919

Key words: numerical mathematics

Preface

This is a book about numerically solving partial differential equations occurring in technical and physical contexts and we (the authors) have set ourselves a more ambitious target than to just talk about the numerics. Our aim is to show the place of numerical solutions in the general modeling process and this must inevitably lead to considerations about modeling itself. Partial differential equations usually are a consequence of applying first principles to a technical or physical problem at hand. That means, that most of the time the physics also have to be taken into account especially for validation of the numerical solution obtained.

This book in other words is especially aimed at engineers and scientists who have 'real world' problems and it will concern itself less with pesky mathematical detail. For the interested reader though, we have included sections on mathematical theory to provide the necessary mathematical background. Since this treatment had to be on the superficial side we have provided further reference to the literature where necessary.

Delft, June 2005

Jos van Kan
Guus Segal
Fred Vermolen

Note to the first edition improvements

In this improved first edition exercises and theory are more separately presented. Furthermore, some parts, such as the parts on boundary fitted coordinates, on coordinate transformation, the treatment of essential boundary conditions for FEM and the solution of non-linear systems of equations, have been rewritten to make them easier to understand.

Newmark-type solvers for the wave equation have been added.

Delft, April 2008

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Fred Vermolen

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