

# **Mathematical Systems Theory**



# **Mathematical Systems Theory**

fourth edition

G.J. Olsder  
J.W. van der Woude  
J.G. Moks  
D. Jeltsema

Faculty of  
Electrical Engineering, Mathematics and Computer Science  
Delft University of Technology

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# Preface

## Fourth edition

The major difference between this fourth edition and its predecessor is the presentation of the material in Chapter 3. The method of linearization of a system described in this chapter has now been restricted to solution-input pairs that are constant in time (equilibrium pairs) and the presentation of the analytical method of solving a linear system is restricted to the linear systems that are time-invariant. We firmly believe that both these restrictions are a great improvement from a didactic point of view. Another important change in this chapter concerns the introductory text, which we believe to be an improvement in the sense that the connection with the previous chapter on modeling is now described more explicitly. A change in Chapter 4 which is worth mentioning is given by the passage where we deal with the duality between the concepts of controllability and observability. We have given a qualitative interpretation of this phenomenon of duality which we believe to be a useful addition to the merely symbolic method using the transposition of matrices.

Delft, November 2011

J.W. van der Woude, J.G. Maks and D. Jeltsema

## Third edition

Compared to the second edition, the presentation of material in this third edition has been changed significantly. For a start, based on feedback by students, certain topics, like linearization, Routh's criterion, interval stability, observer and compensator design, have been discussed in some more detail than in the second edition. Further, in each chapter theorems, lemmas, examples, and so on, are numbered consecutively now, and exercises have been moved towards the end of chapters. Also additional exercises have been included. Finally, errors and typos, found in the second edition, have been corrected. A.A. Stoorvogel and J.G. Maks are greatly acknowledged for their remarks on the second edition. We also thank VSSD for its willingness to publish these notes as a book. We hope that this third edition will be as successful as the previous ones.

Delft, November 2004

G.J. Olsder and J.W. van der Woude

## Second edition

The main changes of this second edition over the first one are (i) the addition of a chapter with MATLAB<sup>®</sup><sup>1</sup> exercises and possible solutions, and (ii) the chapter on ‘Polynomial representations’ in the first edition has been left out. A summary of that chapter now appears as a section in chapter 8. The material within the chapter on ‘Input/output representations’ has been shifted somewhat such that the parts dealing with frequency methods form one section now. Moreover, some exercises have been added and some mistakes have been corrected. I hope that this revised edition will find its way as its predecessor did.

Delft, December 1997

G.J. Olsder

## First edition

These course notes are intended for use at undergraduate level. They are a substantial revision of the course notes used during the academic years 1983-’84 till 1993-’94. The most notable changes are an omission of some abstract system formulations and the addition of new chapters on modelling principles and on polynomial representation of systems. Also changes and additions in the already existing chapters have been made. The main purpose of the revision has been to make the student familiar with some recently developed concepts (such as ‘disturbance rejection’) and to give a more complete overview of the field.

A dilemma for any author of course notes, of which the total contents is limited by the number of teaching hours and the level of the students (and of the author!), is what to include and what not. One extreme choice is to treat a few subjects in depth and not to talk about the other subjects at all. The other extreme is to touch upon all subjects only very briefly. The choice made here is to teach the so-called state space approach in reasonable depth (with theorems and proofs) and to deal with the other approaches more briefly (in general no proofs) and to provide links of these other approaches with the state space approach.

The most essential prerequisites are a working knowledge of matrix manipulations and an elementary knowledge of differential equations. The mathematics student will probably experience these notes as a blend of techniques studied in other (first and second year) courses and as a solid introduction to a new field, viz. that of mathematical system theory, which opens vistas to various fields of application. The text is also of interest to the engineering student, who will, with his background in applications, probably experience these notes as more fundamental. Exercises are interspersed throughout the text; the student should not skip them. Unlike many mathematics texts, these notes contain more exercises (61) than definitions (31) and more examples (56) than theorems (36).

For the preparation of these notes various sources have been consulted. For the first edition such a source was, apart from some of the books mentioned in the bibliography, ‘Inleiding wiskundige systeemtheorie’ by A.J. van der Schaft, Twente University of

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<sup>1</sup>MATLAB is a registered trademark of The MathWorks, Inc.

Technology. For the preparation of these revised notes, also use was made of ‘Course d’Automatique, Commande Linéaire des Systèmes Dynamiques’ by B. d’Andréa-Novel and M. Cohen de Lara, Ecole Nationale Supérieure des Mines de Paris. The contents of Chapter 2 have been prepared by J.W. van der Woude, which is gratefully acknowledged. The author is also grateful to many of his colleagues with whom he had discussions about the contents and who sometimes proposed changes. The figures have been prepared by Mrs T. Tijanova, who also helped with some aspects of the  $\text{\LaTeX}$  document preparation system by means of which these notes have been prepared.

Parallel to this course there are computer lab sessions, based on MATLAB, by means of which the student himself can play with various examples such as to get a better feeling for concepts and for designing systems himself. This lab has been prepared by P. Twaalfhoven and J.G. Braker.

Delft, April 1994

G.J. Olsder

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