Order in space

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PREFACE

This book deals with a very old subject. Many centuries ago some people were already fascinated by polyhedra, and they spent much time in investigating regular spatial structures. In the terminology of polyhedra we, therefore, meet the names of Archimedes, Pythagoras, Plato, Kepler etc.
Today, polyhedra play a role in crystallography, art, architecture, and, in particular, in hobbyistic mathematics: many people are still so much delighted by them, that they cannot leave from constructing, analyzing and playing with polyhedra, and, above all, enjoying them.
This book is the result of several decades of these activities, step by step carried out in spare time. It is intended to stimulate fellow-hobbyists to a further interest in this fascinating subject !

Delft, February 2001 A.K.van der Vegt

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1 INTRODUCTION

1.1 WHAT IS IT ABOUT?

If you ask somebody to mention a polyhedron, the most probable answer will be: a cube. We know the cube as a dice or as a box, but also in a distorted shape, as a rectangular or a skew block. Prisms are known as well: bars with flat faces (when there are four faces, we are again near the cube). Further: pyramids, known from Egypt.

Not all these polyhedra are of importance for this book, but only those which show a pronounced regularity, for example because all vertices and/or all faces are equal and / or regular. The blocks and the pyramids will, therefore, be abandoned, except those which show a distinct regularity, such as a block with square faces (a cube), or a three-sided pyramid with equilateral triangles, a regular tetrahedron.

Already now we have met two wholly regular polyhedra: the cube (hexahedron, R6) and the tetrahedron (R4). As we are about to discover, three other types exist: the octahedron (R8), the dodecahedron (R12) and the icosahedron (R20). These five bodies are already fascinating enough to be studied in detail, each in itself, and in their mutual relations.

But there is more! We can extend our collection in several ways, e.g. allowing concessions to be made on complete regularity (we then arrive at the half-regular or uniform bodies), and also to consider stellated polyhedra. Wholly new worlds are then opened, which, in this book, will be superficially scouted.

First of all a brief survey on what, over the past 25 centuries, has become known on polyhedra.

1.2 AN OLD SUBJECT

2500 years ago (about 520 B.C.), Pythagoras already knew about the existence of three of the five regular polyhedra: he described the cube, the tetrahedron and the dodecahedron.

Plato (about 350 B.C.) knew all five of them, inclusive the octahedron and the icosahedron, and he related them as “cosmic building stones of the world” to the five elements: fire, air, water, earth and “heavenly substance”. Therefore these five bodies

are designated as “Platonic bodies”. Euklides (about 300 B.C.) described them in more detail.
To Archimedes (about 250 B.C.) knowledge on the 13 uniform or Archimedean bodies is attributed by Pappus (500 year later!).
Then, after a very long time, Kepler (1571-1630) (1) arrives at an integrated description of the five Platonic and the 13 Archimedean solids. Kepler also stated that the five Platonic solids are related to the structure of the solar system (besides the Earth, only five planets were known at that time!). Kepler also arrived at the idea that also pentagrams (regular five-sided stars) could lead to regular polyhedra, and he constructed the small and the great stellated dodecahedron.

Two centuries passed before Poinsot (1777-1859) (2) complemented this series of two to the complete collection of wholly regular star-polyhedra.
Subsequently, piece by piece, stellated and other higher-order uniform polyhedra are being published. Each time some more of these are “discovered”, in particular by Pitsch (3), Brueckner (4) and Hess (5) (all end of 19th century).
Coxeter (6) carried out a mathematical analysis of the various polyhedra, and also of the extension to higher dimensions (polytopes).
Finally: The photo-book of Wenninger (7) presents a brief analysis and a brilliant survey of a great number of polyhedra. Moreover. this book provides extensive directions to construct them.

1.3 WHAT ARE POLYHEDRA?

Polyhedra are parts of space, enclosed by flat polygons, such as the cube, which is enclosed by six squares. When we consider the various polyhedra, we shall, therefore, first focus our attention on their faces. But also the vertices are of importance, since at these points a number of faces (at least three) join together and also a number of edges (again three or more). The faces are polygons, the vertices form spatial angles; each of these is characterized by: regular or irregular, number of edges etc. To get some understanding about polyhedra we shall, therefore, first have to consider their faces as well as their vertex figures.

1.4 POLYGONS

A polygon is a flat figure, defined by a closed chain of a number of line segments (the sides) A_1A_2, A_2A_3, A_3A_4,..., $A_{n-1}A_n$, A_nA_1, connecting consecutive pairs of n points A_1, A_2,...A_n (the vertices) (Figure 1.1). For the time being we only consider polygons whose sides do not intersect.

A polygon is called equilateral when all sides are equal, and equiangular when its angles are equal. When a polygon is both equilateral and equiangular, it is a regular polygon. This is the case with the equilateral triangle {3}, the square {4} etc.

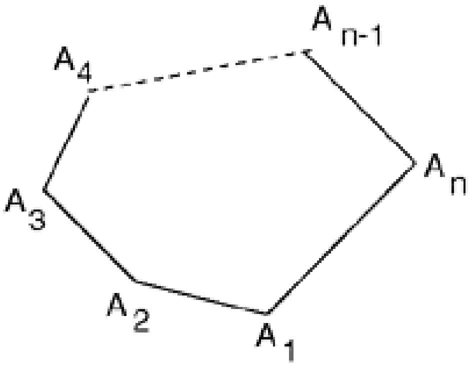


Figure 1.1 Polygon

Equilateral but not regular is, e.g. the rhomb; equiangular but not regular is the rectangle. As a matter of fact there is an infinite number of regular polygons; we designate them as {n}. The angles of {n} can be easily calculated by realizing that an {n} can be divided into n – 2 triangles, each with a sum of angles of 180°, so that each vertex has an angle of 180°·(n – 2)/n. For {3 }, {4 }, {5 }, {6 }, {8 } and {10 } these values are, respectively, 60°, 90° ,180°, 120°, 135° and 144°.
Each {n} has a circum-circle and an in-circle; their radii, r_0 and r_i are related to each other and to the length of the side l as follows:

$$l = 2r_0 \sin(180°/n) = 2r_i \tan(180°/n).$$

The surface area of {n} is:

$$A = (n/4) \cdot l^2 \cdot \text{cotg}(180°/n) = (n/2) \cdot r_0^2 \cdot \sin(360°/n).$$

With increasing *n* this value approaches to the surface area of the circum-circle, πr_0^2

1.5 POLYHEDRAL ANGLES

Polyhedral angles are formed by a number of planes (three or more), which intersect at a point O (Figure 1.2), and which are arranged in such a way that their intersection with another plane not passing through O, forms a polygon (B_1 B_2...B_m). The line segments OB_1, OB_2, …, OB_m are the edges of the multihedral angle; the parts of the planes enclosed by these edges are the sides. It is obvious that the number of edges as well as the number of sides equals m.
The sides are expressed as the angles, formed by adjacent edges (e.g. B_0 O B_1), while the dihedral angles between adjacent sides are called the angles of the polyhedral angle.

A polyhedral angle can, as well as a polygon, be equilateral or equiangular; if it is both, the polyhedral angle is regular. An example is the top of a regular four-sided pyramid which has a square as its base.

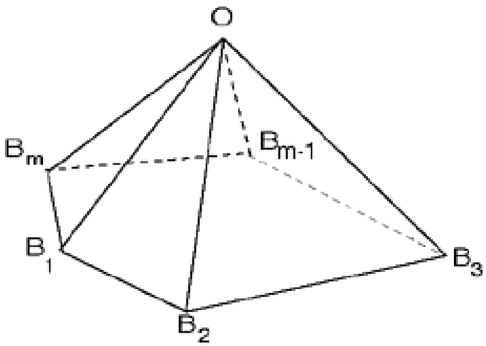


Figure 1.2 Polyhedral angle

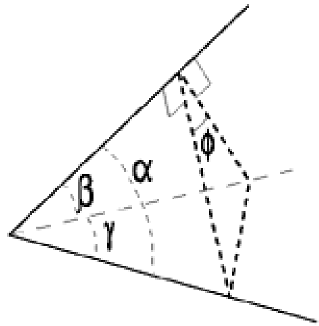


Figure 1.3 Dihedral angle

When the sides of a trihedral angle are α, β and γ, then the angle between the sides α and β is given by (see Figure1.3)

$$\cos\varphi = \frac{\cos\gamma - \cos\alpha \cdot \cos\beta}{\sin\alpha \cdot \sin\beta} \tag{1.1}$$

This can be simply derived by twice applying the rule of cosines.
For an m-hedral angle with $m > 3$ such a general relation can, of course, not be given, since its shape is not uniquely determined by the magnitude of the sides (analogous to an n-gon with $n > 3$). For regular m-hedral angles the dihedral angle between adjacent sides is given by:

$$\cos\varphi = \frac{\cos\alpha - p}{\cos\alpha + 1} \tag{1.2}$$

in which $p = 1 + 2\cdot\cos(360°/m)$, so $p = 0, 1, (\sqrt{5}+1)/2$ and 2 for $m = 3, 4, 5$ and 6 respectively.

1.6 POLYHEDRA

A polyhedron is a solid, enclosed by a number of polygons (faces), which, two by two, have a side in common (edges), while three or more polygons join in common vertices.
Polyhedra can, just as polygons, be equilateral (all faces are identical) or equiangular (all vertices are identical). Now the combination of these is, however, not sufficient to call a polyhedron regular. An example is the disphenoid, which can be constructed

from an irregular triangle by connecting the middles of the sides and folding the it along the connecting lines into a tetrahedron (Figure 1.4).

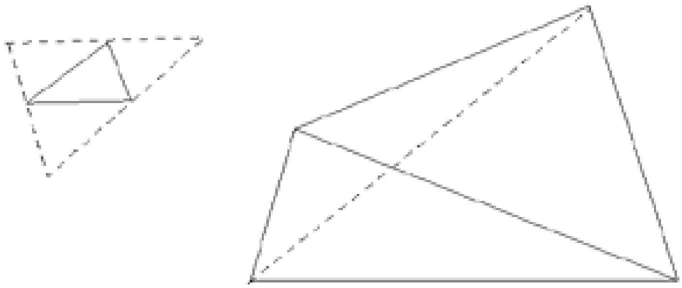


Figure 1.4 Disphenoid

An extra requirement for a polyhedron to be regular, is that the faces are regular polygons; then the vertices are also regular. These regular polyhedra are called "Platonic solids"; as we have already seen, there exist five of them, and they will be described in Chapter 2.

Next to these Platonic polyhedra other ones exist which are less regular, such as the uniform or Archimedean polyhedra. These can be divided into two classes:

- the faces are regular but not equal, while the vertices are identical (first kind),
- the vertices are regular but unequal, while the faces are equal but not regular (second kind).

These polyhedra will be dealt with in Chapters 3 and 4.

Finally: a whole new world is opened when we drop the requirement that the edges of a regular polygon are not allowed to intersect. We then obtain a new series of regular polygons (higher-density polygons such as the five-star, the pentagram), and also new series of regular and uniform polyhedra. These are the subjects of Chapters 5, 6 and 7.

1.7 EDGES, VERTICES AND FACES

Before considering the various polyhedra in more detail, it is useful to find out a general relation between the numbers of vertices, V, faces, F, and edges, E. Such a relation will be very helpful when we analyse the various possibilities to construct a polyhedron.

Euler defined the formula: $V + F = E + 2$, which can easily be verified for, e.g., a cube: $V = 8$, $F = 6$, $E = 12$. This rule can be proven in different ways. One of these is as follows:

To count faces, vertices and edges, we imagine a plane which initially is wholly outside the polyhedron, and which we gradually shift through the polyhedron until it becomes again free at its other side. The plane moves in such a way that its first and its last contact with the polyhedron takes place at a vertex, while, during its trip, it passes only one vertex at the same time. The latter is possible because the plane may be curved in arbitrary directions, without affecting the result.

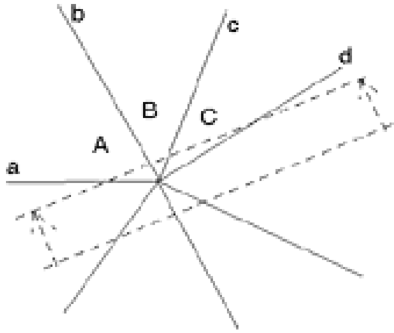


Figure 1.5 Euler's formula

Let us first consider an arbitrary position of the plane, somewhere during its trip through the polyhedron; from this position we shift the plane just enough to pass through a vertex.

Then the plane meets for the first time p edges (a, b, c and d in Figure 1.5), and p – 1 faces (A, B and C). The values of V, F and E are then increased by $\Delta V = 1$, $\Delta F = p - 1$, $\Delta E = p$. So V + F – E is not changed. This holds for all vertices passed, except for the first and for the last one. When the first vertex passed is an m-hedral angle, upon passing it $\Delta V = 1$, $\Delta F = m$, $\Delta E = m$. At the last vertex $\Delta V = 1$, $\Delta F = 0$ and $\Delta R = 0$. In total it appears that:

$$V + F = E + 2 \tag{1.3}$$

Euler's formula is not valid for all kinds of polyhedra. A first condition is: it should be possible to project the polyhedron from a point at the inside onto a sphere round it in a single way, so that, after “blowing it up”, it forms a single sphere. This condition is not met with the higher-density polyhedra, dealt with later on; we will then consider a modified version of Euler's formula.

Another limitation is the requirement that from each vertex every other vertex can be reached by travelling along edges. This condition is, e.g., not met for a polyhedron consisting of a cube, extended on the middle of one of its faces by a smaller cube. In this case V + F = E + 3 , (16 + 11 = 24 + 3).

If a polyhedron is not convex (containing also obtuse dihedral angles) Euler's rule remains valid: it can as well be transformed into a single sphere by “blowing it up”.

2

COMPLETE REGULARITY (PLATONIC SOLIDS)

2.1 GENERAL

A polyhedron is called regular when its faces as well as its vertices are regular and identical. This means that its faces can have three shapes: {3 }, {4 }, or {5 }. {6 } and higher are not possible, since joining three regular 6-gons results in a flat plane. Apparently the sum of the angles of the polygons, meeting at a vertex, should not exceed 360°.

With this condition in mind, we can easily conclude that there are five possibilities, namely at each vertex 3 or 4 or 5 triangles, 3 squares or 3 pentagons. Each of these combinations can, in principle, result in a regular polyhedron. Whether these possibilities really exist, and how the polyhedra look like, has still to be sorted out.

This can be done in the following way: Let us imagine a polyhedron with V vertices, F faces and E edges, in which at each vertex m n-gons meet. Then the number of flat angles equals F · n but also H · m. In this way the edges have also be counted, but twice, since each edge takes part in two flat angles. So:

$$F \cdot n = V \cdot m = 2 \cdot E \qquad (2.1)$$

Combination of these relations with Euler's rule:

$$F + V = E + 2\ ,$$

yields, for known n and m, values for F, V and E. The result is:

$$F = \frac{2m}{n + m - nm/2} \qquad V = \frac{2n}{n + m - nm/2} \qquad E = \frac{n \cdot m}{n + m - nm/2} \qquad (2.2)$$

From these relations it appears, in the first place, that n + m - nm/2 should be positive. This is nothing else than the condition mentioned before that the sum of the angles at each vertex should not exceed 360°. The sum of the angles in an n-gon is namely $(n - 2)\cdot 180°$, so that the condition becomes: $(m/n)\cdot(n - 2)\cdot 180° < 360°$ or:

$$m(n - 2) < 2n \quad \text{or} \quad n + m - nm/2 > 0.$$

We now introduce the quantity "angular deficiency", which means the deficit compared to 360° (a flat plane), when we join the faces into a vertex. This angular deficiency amounts to:

$$360° - 180° \cdot m(n-2)/n = 720° \cdot \frac{n + m - nm/2}{2n} = 720°/V.$$

From this equation it appears that the sum of the angular deficiencies for all vertices amounts to 720° (4π) (which, by the way, also holds for irregular polyhedra). This fact enables us to calculate the number of vertices (if they are equal) from the angular deficiency. For a vertex at which three regular 5-gons meet, the deficiency amounts to 360° – 3·108° = 36° , so V = 720/36 = 20. From form. (2.1) it follows that F/V = m/n = 3/5, so F = 12 and E = 30.

If we now substitute the following combinations of {n ,m} in the expressions for F, V and E: {3 ,3}, {3 ,4}, {3 ,5}, {4 ,3}, and {5 ,3} then we find the following values:

	n	m	F	V	E	name	
{3,3}	3	3	4	4	6	tetrahedron	R4
{3,4}	3	4	8	6	12	octahedron	R8
{3,5}	3	5	20	12	30	icosahedron	R20
{4,3}	4	3	6	8	12	hexahedron (cube)	R6
{5,3}	5	3	12	20	30	dodecahedron	R12

This table presents the well-known five regular or Platonic solids. Figure 2.1 gives a picture of each of them. They can be designated by different notations, e.g. as R4, R8 etc, but also according to the polygons meeting in a vertex. R8 can thus be called (3 3 3 3) and R12 (5 5 5), or also, in shorter notation, {3,4} and {5,3}, respectively. In this book each of these notations will be used for easy connection with other polyhedra (uniform and higher-density) which will be dealt with later on.

Before considering the five Platonic solids separately, it is useful to realize that in the equations for F, V and E, n and m can be interchanged with respect of E, while n in combination with V can be interchanged with the combination of m and F. This is a consequence of the general rule in geometry that in each relation points can be interchanged with planes and vice versa, whereby lines stay equal. This duality is evident from the table: R6 and R8 ({4,3} and {3,4}) are dually related to each other, and also R12 and R20 ({5,3} and {3,5}), while R4 or {3,3} is, by dual exchange, transformed into itself.

All regular polyhedra have a circum-sphere and an in-sphere, in other words: there is a sphere passing through all vertices and one touching all sides.

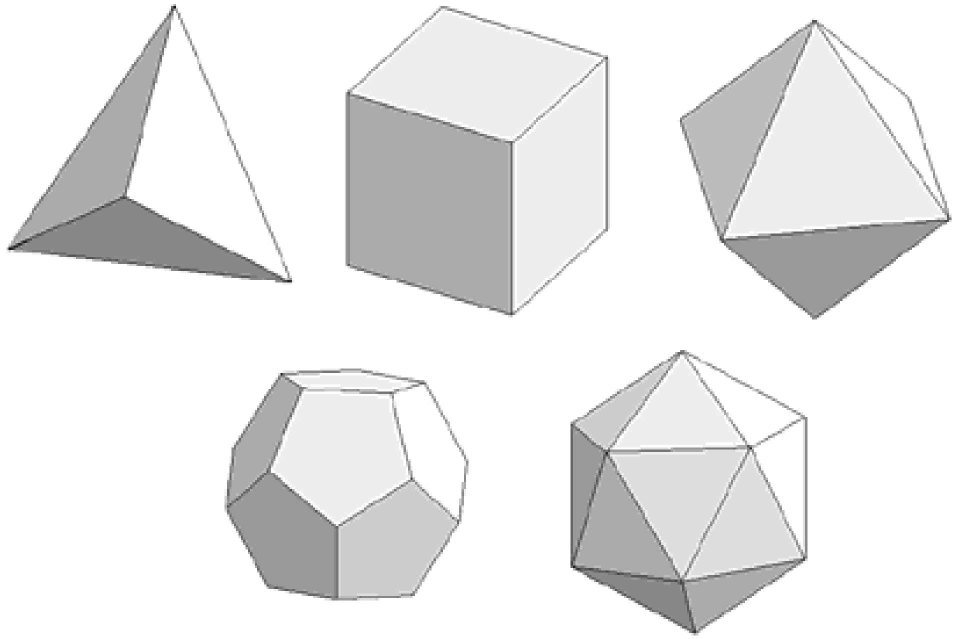


Figure 2.1 The five Platonic solids.

2.2 TETRAHEDRON, R4 OR {3,3} OR (3 3 3)

The regular tetrahedron is bounded by four equilateral triangles, arranged into a three-sided pyramid. The six edges are the only connecting lines between the four vertices, in other words: there are no diagonals in the faces or in the solid.
From formula (1,2) it follows easily that the dihedral angle between the faces (with $\alpha = \beta = \gamma = 60°$) is $\varphi = \arccos(1/3) = 70.53°$. This implies that regular tetrahedra cannot fill space, because then, at least, $360/\varphi$ should be an integer. Later on we shall see that a non-regular tetrahedron exists which is able to fill space.
The regular tetrahedron can be considered not only as a pyramid but also as a prismoid, namely by positioning two opposing edges in parallel planes (Figure 2.2). This arrangement shows that a cross-section with a third parallel plane halfway between the other two, forms a square. The tetrahedron is split up into two equal halves, each bounded by two triangles, two trapezia and a square. It is worth while to construct these two polyhedra from cardboard paper; for a layman who is being asked to join these two to the simplest possible solid, this presents, in most cases, an unexpectedly difficult problem!

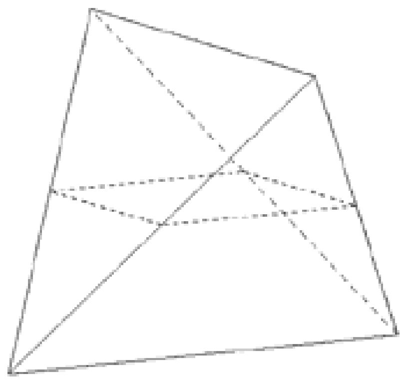


Figure 2.2 Tetrahedron as a prismoid

Tetrahedra can be joined together in many manners into complicated spatial structures; one of the most attractive possibilities is to string them into a twisted tube; the edges at the outside form three spirals (Figure 2.3).

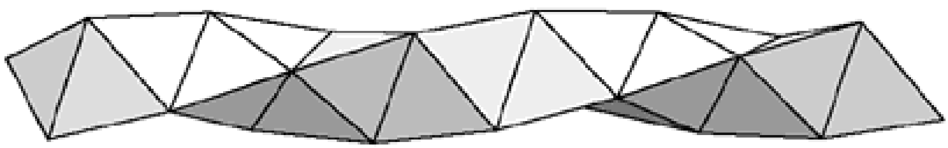


Figure 2.3 Spiralled tube of tetrahedra

2.3 HEXAHEDRON (CUBE), R6 OR {4 ,3} OR (4 4 4)

The cube is by far the best-known regular polyhedron, probably because all angles, between edges as well as between faces, have the convenient value of 90°. Moreover it is able to fill space, with eight cubes joining in a vertex.

The cube looks a bit more complicated when two opposite vertices are considered as top and bottom, while one of the spatial diagonals is vertical. Cross-sections with horizontal planes are then triangles or hexagons, with a regular hexagon halfway (Figure 2.4).

By careful inspection of the cube in this position, it appears to be possible to drill a square hole through the cube in such a way that an equal cube, and even a slightly bigger one, can pass through this hole!!

The cube, placed on a vertex, is incidentally applied in architecture (in The Netherlands near Railway Station Rotterdam-Blaak).

When we draw the diagonals in the faces of a cube, it appears that two R4's can be fitted into an R6 (Figure 2.5) in such a way that the vertices of the tetrahedra ($2 \cdot 4$) coincide with the vertices of the cube. The two tetrahedra partially penetrate each other and have as a common core an R8 which fits into the R6 (see § 2.4). The two tetrahedra form an eight-pointed star ("stella octangula" of Kepler, Figure 2.6).

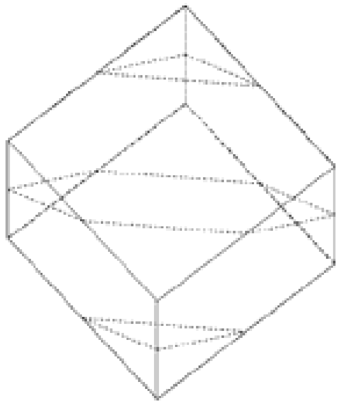


Figure 2.4 Cube in another position

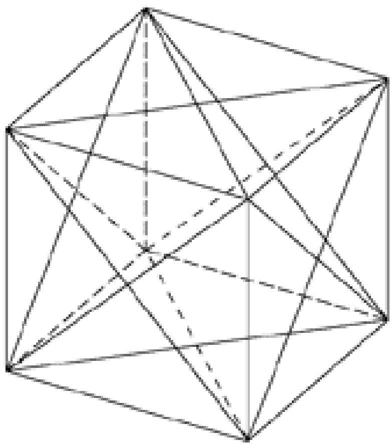


Figure 2.5 Two tetrahedra in a cube

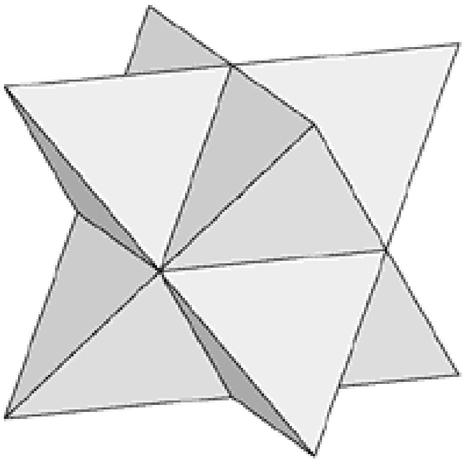


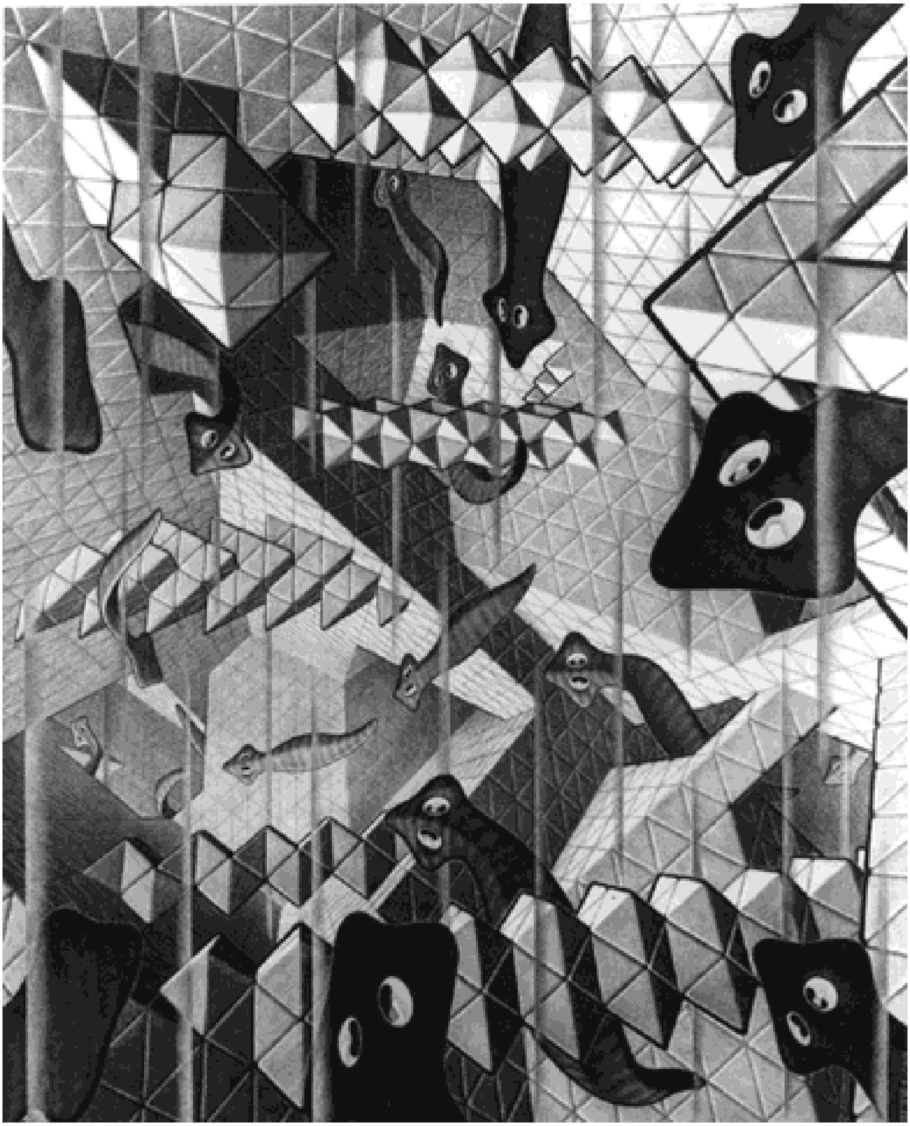
Figure 2.6 The Kepler-star

2.4 OCTAHEDRON, R8 OR {3,4} OR (3 3 3 3)

The regular octahedron can easiest be visualized as built up from two regular four-sided pyramids with a square as common base. Its faces are equilateral triangles, and it contains three square cross-sections since all six vertices are equivalent.

The octahedron can also be considered as a prismoid, in which two opposing faces are situated in two parallel planes (Figure 2.7). Just as with the cube, the cross-section with a parallel plane half-way, forms a regular hexagon.

From formula 1.1 it follows that the dihedral angles between adjacent faces are given by $\cos\varphi = -1/3$, so $\varphi = 109.47°$. It thus appears that the dihedral angles of tetrahedron and octahedron are each other's complements. The octahedron in Figure 2.7 can, therefore, be complemented to a parallelepiped by placing a tetrahedron at its top and at its bottom. With a combination of octahedra and tetrahedra space can be filled, just as with cubes. The space obtained this way is, however, strangely skew (Escher: "Platwormen").



Platwormen [Planaria], © 1959 M.C. Escher / Cordon Art – Baarn – Holland.

In § 2.1 the duality relation between R6 and R8 has already been mentioned. This relation is also evident from the fact that the midpoints of the faces of R8 form the vertices of R6 and vice versa. Moreover, the close relationship makes it possible to construct a compound, in which an R6 and an R8 are concentric while the midpoints of their edges coincide. The rhomb-dodecahedron and the cubo-octahedron (see Chapter 3) owe their existence to this fact.

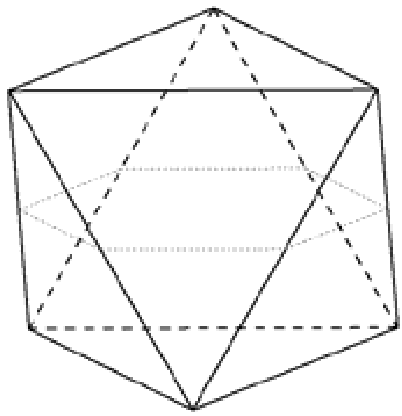


Figure 2.7 Octahedron as a prismoid

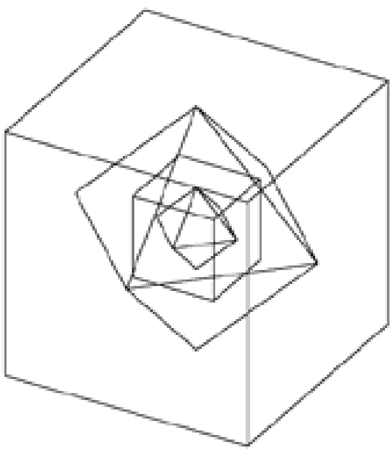


Figure 2.8 Duality of R6 and R8

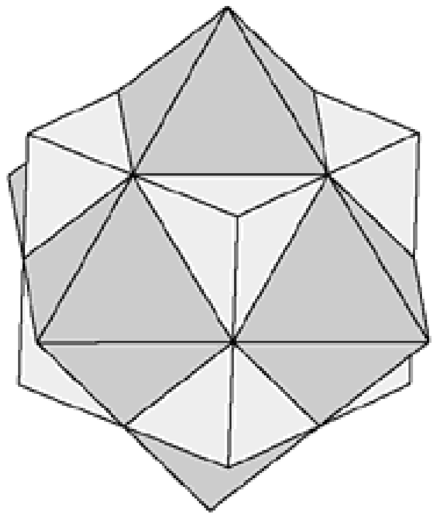


Figure 2.9 Compound of R6 and R8

The relationship between the tetrahedron R4 and the octahedron R8 has already appeared in 2.3: R8 can be fitted into an R4 in such a way that four of its eight faces are coplanar with the faces of R4; the vertices are situated at the midpoints of the edges of the R4.

2.5 DODECAHEDRON, R12 OR {5,3} OR (5 5 5)

The geometry of the dodecahedron is more complicated than we have met so far with the other regular polyhedra, since with this solid a pentagon plays a role for the first time.

Two of the twelve pentagons, of which the R12 is composed, can have a special role, namely as top face and as bottom face. The remaining ten faces can be split-up into two series of five, each of which is adjacent to the top- or to the bottom face. The separation between these two series forms a non-flat decagon (Figure 2.10).

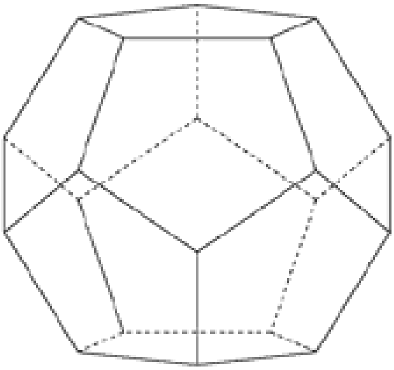


Figure 2.10 The dodecahedron

As remarked before, the dodecahedron is closely related to the icosahedron, R20; this relationship will be further dealt with in 2.6. But also with respect to the cube R6 there exists a surprising relation, even a double one, since a cube can be fitted into as well as around a dodecahedron (Figures 2.11 and 2.12).

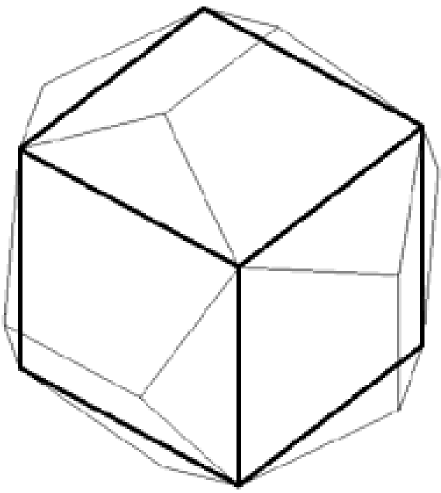


Figure 2.11 R6 in R12

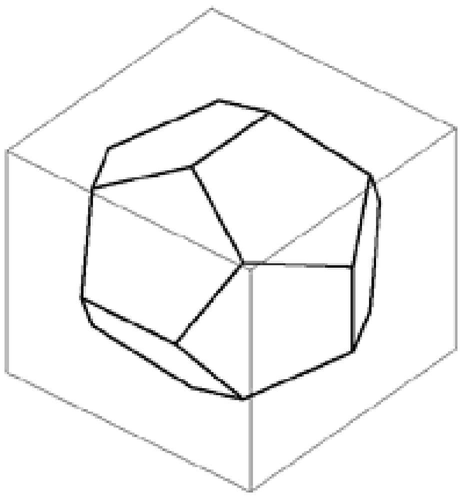


Figure 2.12 R12 in R6

In Figure 2.11 a cube is positioned within a dodecahedron; the vertices of the cube coincide with 8 of the 20 vertices of the dodecahedron and the edges of the cube are formed by the diagonals in the faces of R12. Since the 12 faces of R12 have in total 60 diagonals, 1/5 of the diagonals and 2/5 of the vertices are "used". It is easy to imagine that in this way five cubes can be accommodated into a dodecahedron; each vertex of the R12 then participates in two cubes. This is shown in Figure 2.13. The five cubes form a compound with 30 squares which intersect in a rather complicated way, but which show a beautiful regular pattern of stars.

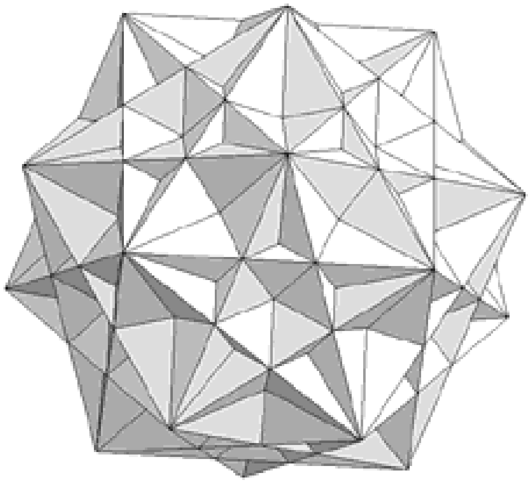


Figure 2.13 Five cubes, fitting into a dodecahedron

Around an R12 a cube can also be constructed in a simple way, as demonstrated in Figure 2.12. Six of the 30 edges of R12, and, therefore, 12 of the 20 vertices, are situated in the faces of the cube. The edge of the cube, expressed in the edge of R12, is $(3 + \sqrt{5})/2$; the edge of R12 is $(3 - \sqrt{5})/2$ times the edge of R6. The eight remaining vertices, not situated in the faces of the R6, form the vertices of the in-cube mentioned above. All of this makes it quite easy to construct a projection of the R12: starting from a cube with unity edges one draws the six edges of R12 with a length of $(3 - \sqrt{5})/2$ and the vertices of a smaller cube with an edge length of $(\sqrt{5} - 1)/2$. It will be clear that also in this way not only one but five cubes can enclose the R12. These five form the same penta-hexahedron as the one represented in Figure 2.13.

In § 2.3 we saw that the vertices of a cube coincide with those of two enclosed tetrahedra. It follows that into the dodecahedron also tetrahedra can be fitted; we can construct two series of each five tetrahedra, forming each other's mirror image, but also a ten-fold compound (deca-tetrahedron) containing both series.

2.6 ICOSAHEDRON, R20 OR {3,5} OR (3 3 3 3 3)

The simplest way to visualize a regular icosahedron is by thinking it as being built-up from two five-sided pyramids with a disk in between consisting of a ring of 10 triangles (see Figure 2.14). In this way an icosahedron fits exactly into a dodecahedron which rests on one of its faces as a base. (see Figure 2.15). The vertices of the R20 are in the midpoints of the faces of R12 (they are dually related to each other).

However, a different position offers more perspective, namely the one in which the top and the bottom of the solid are not formed by a vertex but by an edge. These to parallel edges together form a vertical plane. It then appears that a second pair of edges also lies in a vertical plane and a third pair in a horizontal plane, in other words: R20 can be fitted into a cube! This is demonstrated in Figure 2.16. Six of the 30 edges and all of the twelve vertices are situated in the faces of the cube, the edge of which has a length of $(\sqrt{5} + 1)/2$ times that of the icosahedron. This offers a method of constructing the R20, even easier than for the R12: we only need to draw six mid-parallels in the faces of a cube, give the edges the proper length and connect the vertices in the proper order.

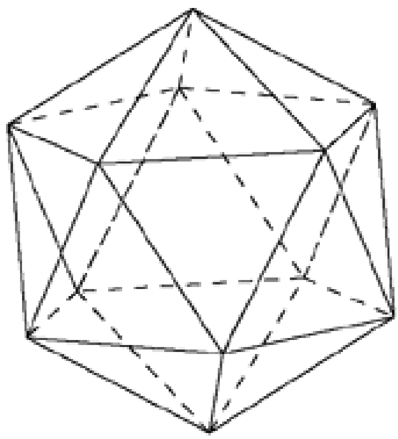


Figure 2.14 The icosahedron R20

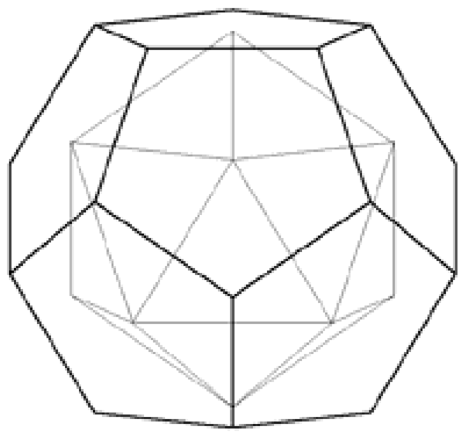


Figure 2.15 R20 in R12

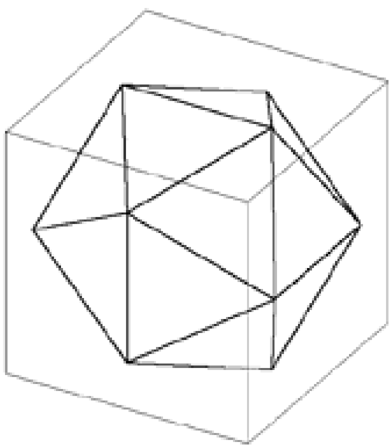


Figure 2.16 Icosahedron in a cube

An attractive consequence is that an icosahedron can evolve from an octahedron within a cube. We start off drawing an R8 by connecting the midpoints of the faces of the cube and we let these six vertices grow into edges as indicated in Figure 2.17 (in which the cube is not shown). The 12 edges of the R8 then extend to faces, which,

together with the original 8 ones, will form the 20 faces of the R20. During this growth process the triangles forming the original R8 shrink and rotate around their axis; the faces are shifted in the outside direction, but remain parallel to their initial position, perpendicular to a spatial diagonal of the cube. Eventually an R20 is formed with 8 of its faces still parallel to the faces of the original R8, in other words: these faces still form an R8 which encloses the R20. All vertices of the R20 lie on the edges of this R8, and 8 of its faces are coplanar with the octahedron faces. The relation between R8 and R20 is, therefore, a very narrow one!
It appears from Figure 2.17 that one and the same face of R20 can take part in a face of R8 in two different ways; these two possibilities represent two of the five ways in which an R8 can enclose an R20. The five octahedra together form a compound: a penta-octahedron.

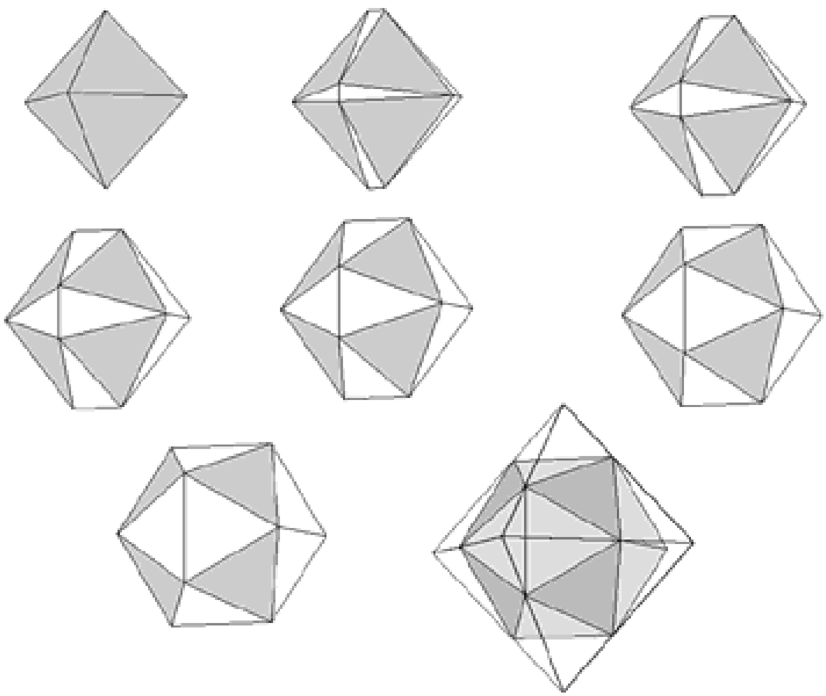


Figure 2.17 Octahedron, evolving into an icosahedron

2.7 GEOMETRICAL CONSTANTS OF THE PLATONIC SOLIDS

In the foregoing we have seen that the Platonic solids can, in various ways, be fitted into and around each other. Utilizing the possibilities of fitting within a cube, we can, very easily, express the vertices of the various species in rectangular coordinates (x,y,z). Their midpoints always coincide with the origin of the coordinate system.

	(x,y,z)	edge length
R4	(1,1,1), (–1, –1,1), (1, –1, –1), (–1,1, –1)	$2\sqrt{2}$
R6	(±1,±1,±1)	2
R8	(±1,0,0), (0,±1,0), (0,0,±1)	$\sqrt{2}$
R12	(±a,±a,±a), (±a^2,0,±1), (0,±1,±a^2), (±1,± a^2,0)	a^2
R20	(±a,0,±1), (0,±1,±a), (±1,±a,0)	a

in which $a = (\sqrt{5} - 1)/2$ and $a^2 = (3 - \sqrt{5})/2$.

Of further importance are the total surface areas and the volumes of the solids. The first can be calculated with simple plane geometry. The volumes can be found by dividing the polyhedron into a number of pyramids, all with their top in the centre and with the faces as bases. The height of such a pyramid is the radius of the in-sphere, r_i, and can be calculated, by applying analytical geometry, as the distance from the origin to a plane through three of the vertices of the relevant face.The radius of the circum-sphere, r_0, is the distance from the origin to any vertex. These calculations result in the following table, giving various quantities, expressed in the edge length as a unit. Moreover, the dihedral angles φ between the faces are given.

	R4	R6	R8	R12	R20	sphere
r_0	$(\sqrt{6})/4$	$(\sqrt{3})/2$	$(\sqrt{2})/2$	$(\sqrt{3})(\sqrt{5}+1)/4$	$(\sqrt{10+2\sqrt{5}})/4$	1
r_i	$(\sqrt{6})/12$	1/2	$(\sqrt{6})/6$	$(\sqrt{10}(\sqrt{25+11\sqrt{5}})/20$	$(\sqrt{3}(3+\sqrt{5})/12$	1
r_0/r_i	3	$\sqrt{3} = 1{,}732$	$\sqrt{3} = 1{,}732$	$\sqrt{15-6\sqrt{5}} = 1{,}258$	$\sqrt{15-6\sqrt{5}} = 1{,}258$	1
surface area A	$\sqrt{3}$	6	$2\sqrt{3}$	$(3\sqrt{5})\sqrt{5+2\sqrt{5}}$	$5\sqrt{3}$	4π
volume V	$(\sqrt{2})/12$	1	$(\sqrt{2})/3$	$(15+7\sqrt{5})/4$	$5(3+\sqrt{5})/12$	$4\pi/3$
A/V	14.70	6	7.35	5,312	5,184	4,836
cos φ	1/3	0	–1/3	$-(\sqrt{5})/5$	$-(\sqrt{5})/3$	(–1)
φ	70,53°	90°	109,47°	116,57°	138,19°	(180°)

From the values given in the table it appears that with increasing number of faces the polyhedra approach the sphere more closely. The dihedral angles increase, though the largest one (R20) is still far removed from 180°. The ratio between the radii of circum-sphere and in-sphere decreases; for dually related polyhedra this ratio is the same, which is also easily understood when we consider a series of dually related solids which enclose each other: the circum-sphere of one is always the in-sphere of the other. Finally we can consider the surface areas for a given volume: for this purpose we have to reduce the volume to unity; it appears that, with the same volume, R20 has an only 7% greater surface area than the sphere (see also § 3.18).

2.8 TOPOLOGICAL PROJECTIONS

The topological projection is an easy aid to investigate structures and characteristics of polyhedra. All faces, edges and vertices are pictured in a flat plane in their mutual relation, albeit, of course, in a distorted way.

A tetrahedron can be represented in three different ways (Figure 2.18). In the first figure the outer triangle represents one of its faces; in the second the fourth vertex is pictured three times; in the third the protruding lines together form the sixth edge.

Figures 2.19, 2.20. 2.21 and 2.22 give some possible topological projections of the other Platonic solids; in Figure 2.21 one of the inscribed cubes is indicated with dashed lines.

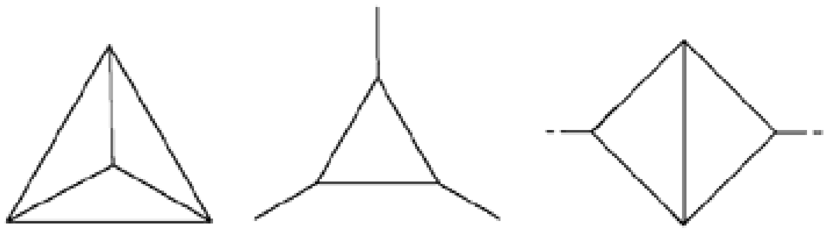


Figure 2.18 Topological projections for R4

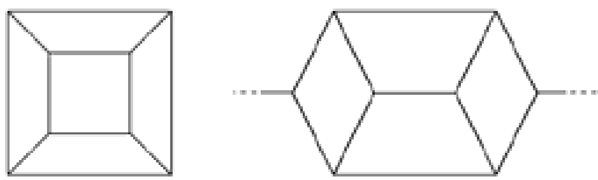


Figure 2.19 also for R6

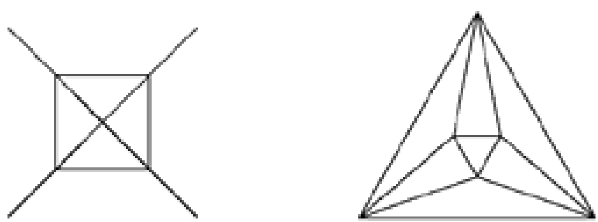


Figure 2.20 also for R8

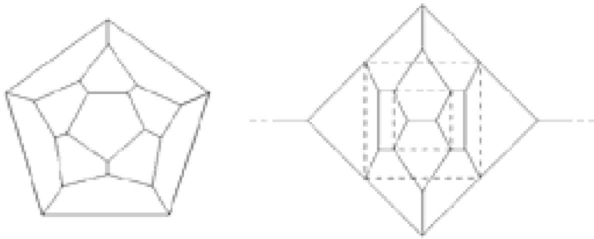


Figure 2.21 also for R12

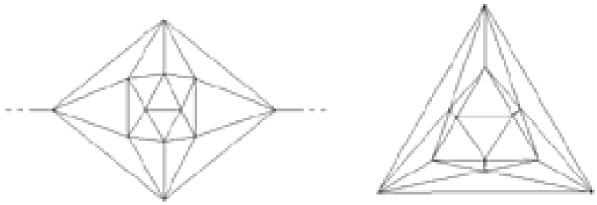


Figure 2.22 also for R20

3

SEMIREGULARITY (ARCHIMEDEAN OR UNIFORM POLYHEDRA)

3.1 GENERAL

In Chapter 1 the definition of uniform polyhedra, also denoted as Archimedean solids, has been given. They are polyhedra with regular polygons as faces, which are not equal (e.g. triangles next to squares etc.), while their vertices are equal but not regular. These are the uniform polyhedra of the first kind.
According to the principle of duality the definition can can be inverted: we then obtain uniform polyhedra of the second kind: the vertices are regular but not equal; the faces are equal but irregular.
The duality between the first and the second kind is already apparent from these definitions, but can also be expressed as follows: When we bring a sphere through the vertices of a U1 (uniform of the first kind), which is possible for every U1, and we construct tangent planes at this sphere at each vertex, then these planes form a U2 (uniform polyhedron of the second kind), which is dually related to the U1. The other way round: the in-sphere in a U2 (which always exists) touches the faces in points which form the vertices of the related U1.
In this chapter we shall look at the uniform polyhedra of the first kind. First of all we have to sort out the possibilities to join various types of regular polygons into a polyhedron.

3.2 ANALYSIS

We consider a single vertex of the Archimedean solid (all vertices are equal), and we assume that in this vertex m_1 n_1-gons, m_2 n_2-gons etc. meet. The number of vertices is V, the number of faces $F = F_1 + F_2 + ... = \sum F_i$, in which F_i is the number of n_i-gons. The calculation of V proceeds along the same lines as for the Platonic solids: the number of flat angles in the n_i-gons is $F_i \cdot n_1$ but also $V \cdot m_i$, so that $F_i = V \cdot (m_i/n_i)$. Moreover, the total number of flat angles is $\sum F \cdot n_i = V \cdot \sum m_i = 2R$ since each edge is counted twice. Combination of these relations with Euler's formula: $\sum F_i + V = E + 2$ results in:

$$V = 4/[2 - \sum[(n-2)(m_i/n_i)] \quad (3.1)$$

This relation allows us to calculate the F_i's with $F_i = V\cdot(m_i/n_i)$ and E with $E = F + V - 2$.

The value of V can also be calculated from the angular deficiencies, in the same way as we did for the Platonic solids, with: $V = 720° / (360° - \sum \alpha_i)$, which, of course, gives the same result.

All uniform polyhedra of the first kind can now be found by systematically trying out combinations of n and m. An example: $m_1 = 2$, $n_1 = 3$, $m_2 = 2$, $n_2 = 4$ (two triangles and two squares form a vertex) results in: $V = 12$, $F_1 = 8$, $F_2 = 6$, $E = 24$, which means a solid, bounded by 8 triangles and 6 squares, with 12 vertices and 24 edges. For this solid we shall use the notation: (3 4 3 4).

For various combinations of n and m the formula gives a non-integer value of V, such as e.g. $m_1 = 2$, $n_1 = 3$, $m_2 = 1$, $n_2 = 4$, (3 3 4): $V = 4.8$; this arrangement does, therefore, not represent an existing polyhedron.

There are, however, also combinations which, though resulting in an integer value of V, do not yield a possible polyhedron, such as $m_1 = 2$, $n_1 = 5$, $m_2 = 1$, $n_2 = 6$: (5 5 6). For this combination we find: $V = 30$, $F_1 = 12$, $F_2 = 5$, $E = 45$. This polyhedron, however, does not exist; when we try to draw its topological projection, we start off with two hexagons, each meeting two pentagons in three of its vertices, but we soon discover that this leads to two (5 5 5) vertices (see Figure 3.1). From further consideration of this and similar situations it appears that for topological reasons a second condition for the existence of an Archimedean solid can be formulated: an odd n-gon should, in each of its corners, be bounded by mutually equal neighbours (such as with (3 4 5 4)), unless in the same vertex two or more extra polygons of its kind meet (such as (3 3 3 3 4)). Apparently (5 5 6) does not meet this condition; moreover it appears that the combination (3 4 3 4) considered before is adequate, but not (3 3 4 4).

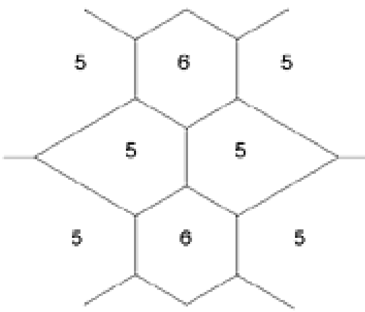


Fig.3.1 "Impossible" polyhedron (5 5 6)

In order to compose the complete list of Archimedean solids, many combinations have to be tried out and analysed (which is a fascinating job!). The final result is given in the Table; it contains 13 separate and two infinite series of uniform polyhedra. In both of the series n can take all values of 3 and higher; the first series contains the Archimedean prisms, the second the antiprisms.

notation	faces	number of faces vertices edges		
		F	V	E
(3 4 3 4)	8{3} + 6{4}	14	12	24
(3 6 6)	4{3} + 4{6}	8	12	18
(4 6 6)	6{4} + 8{6}	14	24	36
(3 8 8)	8{3} + 6{8}	14	24	36
(3 4 4 4)	8{3} + 18{4	26	24	48
(4 6 8)	12{4} + 8{6} + 6{8}	26	48	72
(3 5 3 5)	20{3} + 12{5}	32	30	60
(5 6 6)	12{5} + 20{6}	32	60	90
(3 10 10)	20{3} + 12{10}	32	60	90
(3 4 5 4)	20{3} + 30{4} + 12{5}	62	60	120
(4 6 10)	30{4} + 20{6 } + 12{10}	62	120	180
(3 3 3 3 4)	32{3} + 6{4}	38	24	60
(3 3 3 3 5)	80{3} + 12{5}	92	60	150
(4 4 n)	n{4 } + 2{n }	2+n	2n	3n
(3 3 3 n)	2n{3 } + 2{n }	2+2n	2n	4n

3.3 ARCHIMEDEAN PRISMS (4 4 n)

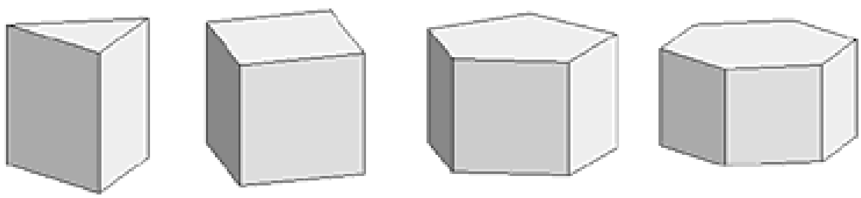


Figure 3.2 Archimedean prisms

The Archimedean prisms are bounded by two n-gons, located in parallel planes, and n squares in planes perpendicular to these planes (Figure 3.2). The simplest Archimedean prism is (4 4 3), the next, (4 4 4) is a Platonic solid, namely the cube. With increasing n the prism becomes flatter and flatter and approaches to a thin cylindrical disk, contrary to the real Archimedean solids which, with increasing number of faces, more and more approach to a sphere. These solids can, therefore, be considered as degenerations of the uniform polyhedra.

3.4 ARCHIMEDEAN ANTIPRISMS (3 3 3 n)

The (3 3 3 n), the Archimedean antiprism, has a more interesting shape than the prism of the previous section; though it is as well bounded by two n-gons in parallel planes, these n-gons are rotated with respect to each others by an angle of 180°/n. The first member of this family is (3 3 3 3), the regular octahedron. the next, (3 3 3 4) is composed of two squares and eight triangles; then we find (3 3 3 5) as the centre part of {3 ,5}, the regular icosahedron (see Figure 3.3). Also the antiprisms approach, with increasing n, more and more to a flat disk, and, therefore, their shape becomes less and less interesting.

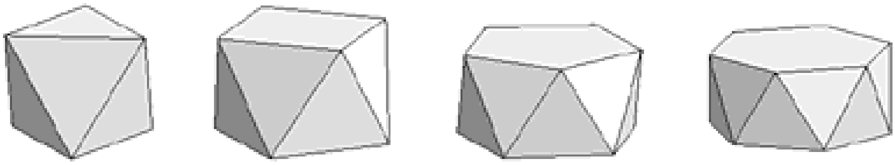


Figure 3.3 Archimedean antiprisms

3.5 THE CUBO-OCTAHEDRON (3 4 3 4)

This polyhedron, together with the (3 5 3 5), which will be dealt with later on, takes a special place in the series of Archimedean solids; it is, namely, the common core of a {3 ,4} (octahedron) and a {4 ,3} (cube), which penetrate each other in such a way that all edges intersect in pairs. From this fact its name is derived: "cubo-octahedron" (see Figure 3.4).

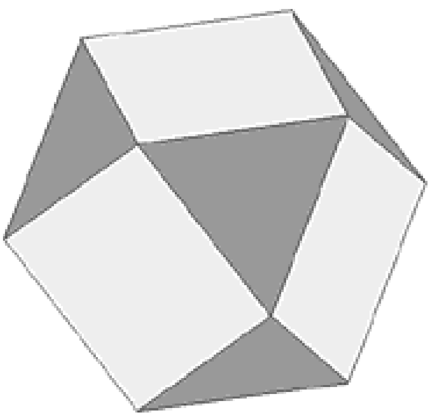


Figure 3.4 (3 4 3 4)

The twelve vertices of the (3 4 3 4) lie at the midpoints of the edges of a cube, and also of a regular octahedron.. A similar situation is found with the (3 5 3 5); these two polyhedra are, therefore, designated as “quasi-regular polyhedra”.
The coordinates of (3 4 3 4) can be easily derived from those of a cube with edge-length = 2, namely (0 ±1 ±1) cycl (cycl means: next to (a b c) also (b c a) and (c a b)); the length of the edge is then $l = \sqrt{2}$.

3.6 THE TRUNCATED TETRAHEDRON (3 6 6)

The (3 6 6) can be considered as a regular tetrahedron {3 ,3} with its four points cut-off in such a way that the remaining parts of the four faces are regular hexagons. The four cutting planes form regular triangles (Figure 3.5). Starting off from a tetrahedron which fits in a cube with edge length = 6, the coordinates of the vertices can easily be represented as follows: (±1 ±1 ±3) cycl with the additional condition: $x \cdot y \cdot z > 0$. The length of the edge is then $l = 2\sqrt{2}$.

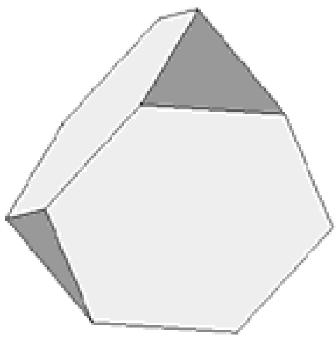


Figure 3.5 (3 6 6)

When the planes which cut off the corners are further shifted towards the midpoint until the hexagon is transformed into a triangle (analogous to the way in which (3 4 3 4) is obtained by truncation of {3 ,4}, the result is a {3,3}.

3.7 THE TRUNCATED OCTAHEDRON (4 6 6)

In the same way as (3 6 6) is formed from {3 ,3}, the (4 6 6) can be obtained from {3 ,4} by truncation; here also the triangles are truncated into hexagons (Figure 3.6). Continued truncation op to the midpoints of the edges results in the (3 4 3 4).
When the coordinates of the initial {3 ,4} are (0 0 ±3) cycl, then those of the (4 6 6) are: (0 ±1 ±2) cycl and the edge length is $l = \sqrt{2}$.
The (4 6 6) possesses a remarkable property, namely that it can be piled up to fill the space completely. This is possible for only a small number of polyhedra, such as the cube, some of the prisms and the ((3 4 3 4)) of the second-kind Archimedean solids (see § 4.7).

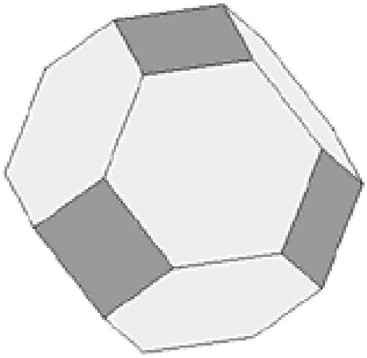


Figure 3.6 (4 6 6)

3.8 THE TRUNCATED CUBE (3 8 8)

The (3 8 8) is a truncated cube {4 ,3}, which is obtained when we cut the cube by eight planes perpendicular to its diagonals (which thus form a {3 ,4}), until the squares are transformed into regular octagons as shown in Figure 3.7 (by further truncation the (3 4 3 4) would be reached). The coordinates of the 24 vertices are: (±c ±1 ±1) cycl, with $c = \sqrt{2} - 1$. In this case the edge length of the original cube is 2, while c follows from the condition that the edges of the octagon are equal, namely $2(\sqrt{2} - 1)$.

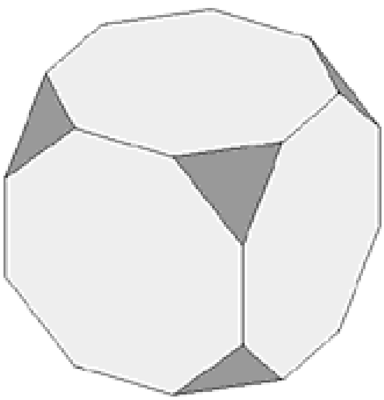


Figure 3.7 (3 8 8)

The truncations dealt with in this section and in the previous one are illustrated in Figure 3.8, in which a cube is truncated to an octahedron via a (3 8 8), a (3 4 3 4) and a (4 6 6), respectively.

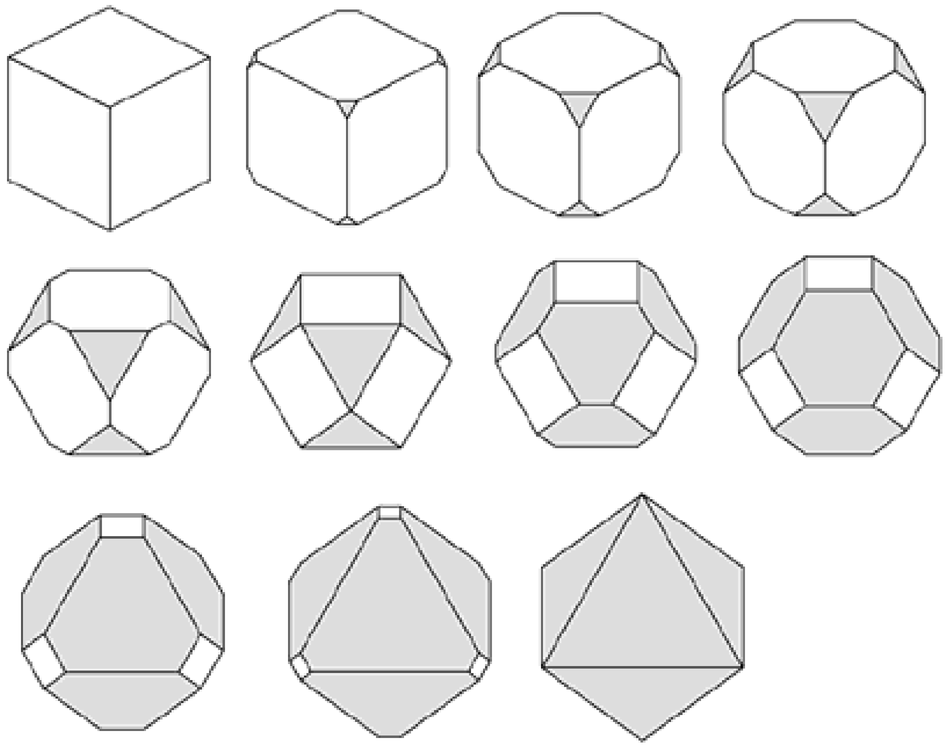


Figure 3.8 Transformation by truncation

3.9 THE RHOMB-CUBO-OCTAHEDRON (3 4 4 4)

This solid is bounded by 8 triangles and 18 squares (Figure 3.9), but can be more easily analysed when we split up the 18 squares into a group of 6 and one of 12. With its triangles it fits namely into an octahedron {3 ,4}, and with 6 of its squares into a cube {4 ,3}. The other 12 squares are then more or less related to the edges of the cube or of the octahedron.

The (3 4 4 4) can originate from a cube by truncation along its edges; the edges are then transformed into rectangles (Figure 3,10). The vertices of the cube thereby extend to triangles. When the position of the cutting planes is properly chosen, the 12 rectangles become squares. The same procedure can be carried out with a regular octahedron, with the same result.

The cross section of (3 4 4 4) with a horizontal plane through its midpoint is a regular octagon, just as with the two other coordinate planes. Thus it is possible to cut an Archimedean 8-sided prism from the (3 4 4 4) in three different ways.

The coordinates of the 24 vertices are: (±1 ±1 ±a) cycl, with $a = \sqrt{2} + 1$ = half the edge length of the circum-cube. The edge length of the polygon is $l = 2$.

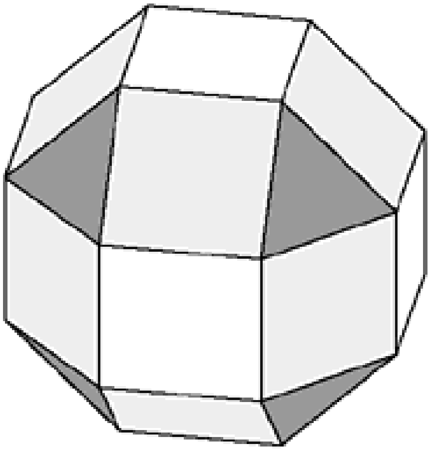


Figure 3.9 (3 4 4 4)

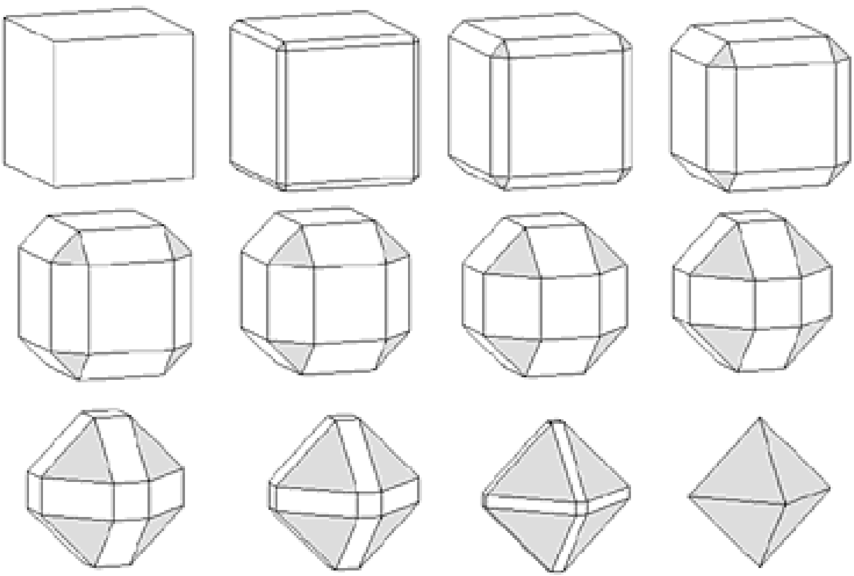


Figure 3.10 Transformation cube - (3 4 4 4) - octahedron

This Archimedean solid has a remarkable variant, namely the solid which results from a rotation of the upper part, which rests on the 8-sided prism, over an angle of 45°. This solid is also a (3 4 4 4): in each vertex three triangles and a square meet. The relation to the {4,3} and the {3,4} is, however disturbed, and there is only one symmetry plane left.

3.10 TRUNCATED CUBOCTAHEDRON (4 6 8)

The (4 6 8) is, in this series, the first uniform polyhedron which has three different types of faces. This solid is again related to the cube and the octahedron: its six octagons coincide with the faces of a cube, its eight hexagons with those of an octahedron, while we can imagine its 12 squares as resulting from "squaring" of the edges of either cube or octahedron (Figure 3 .11)

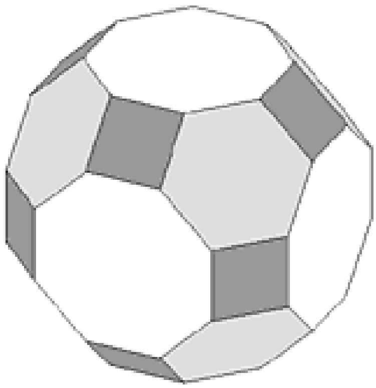


Figure 3.11 (4 6 8)

At a first sight it seems as if the (4 6 8) could be formed by truncating the vertices of a (3 4 3 4) (hence its name); however, it appears that such a truncation can never lead to square faces, but to rectangles only. Therefore, after truncation, the faces have to be shifted over some distance to obtain a correct (4 6 8). A better indication is, therefore : "rhombitruncated cuboctahedron".

The coordinates of the 48 vertices are: (±1 ±a ±b) cycl and (±1 ±b ±a) cycl, in which a = 1 + √2 and b = 1 + 2√2 = half the edge length of the circum-cube; the edge length of the (4 6 8) is then $l = 2$.

3.11 THE ICOSI-DODECAHEDRON (3 5 3 5)

As already remarked when we dealt with (3 4 3 4), the (3 5 3 5) or icosi-dodecahedron is the second solid which is called quasi-regular. It is related to the {3,5} and the {5,3} in the same way as (3 4 3 4) is related to {3,4} and {4,3}. It can originate from {3,5} as well as from {5,3} by truncation of vertices up to the midpoints of the edges (Figure 3.12)

Its vertices are, therefore, situated at the midpoints of the 30 edges of an R20 or an R12. Their coordinates, when starting from an R12 enclosed by a cube, are: (0 0 ±2) cycl (6 vertices of an R8), and (±1 ±a ±(a+1)) cycl (8·3 vertices of triangles within a bigger R8), with a = (√ – 5)/2. Its edge length is then $l = \sqrt{5} - 1$.

The (3 5 3 5) thus fits into all five Platonic solids; with four of them (R4, R8, R12 and R20), it has a face in common with all available faces, while the fifth (R6) contains in each of its faces a vertex of (3 5 3 5).

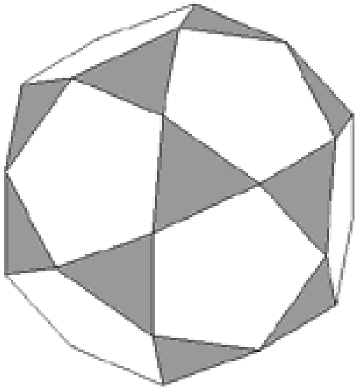


Figure 3.12 (3 5 3 5)

3.12 THE TRUNCATED ICOSAHEDRON (5 6 6)

The (5 6 6) can be thought to originate from truncation of the vertices of an icosahedron {3,5}, in such a way that its triangular faces are transformed into regular hexagons. Also this polyhedron can be fitted into all five Platonic solids. (Figure 3.13).

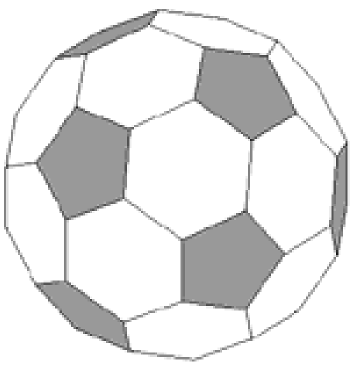


Figure 3.13 (5 6 6)

The coordinates of the 60 vertices are as follows:

(0 ±3 ±a) cycl, the 12 vertices in the faces of the circum-cube,

(±1 ±2a ±(2+a)) cycl, the 24 vertices adjacent to the foregoing series;

(±2 ±a ±(2a+1)) cycl, the remaining 24 vertices,

in which a = (√5 – 1)/2; the edge length is then l = 2a.

This polyhedron has become known in several ways. As a football it can be seen daily on the TV, with black pentagons and white hexagons (though the (3 5 3 5) is also being used). Since a few years it has become a highly interesting polyhedron for chemists, who succeeded in synthesizing C_{60} molecules; the 60 carbon atoms are situated at the vertices of a (5 6 6), and are held together by single and double bonds along the edges of the pentagons and the common edges of the hexagons, respectively. The molecule was called: "Buckminster-fullerene", (Bf), after the architect R. Buckminster Fuller, who, in 1954 patented a dome structure on the basis of this polyhedron. More of these kind of structures are possible by "diluting" it with an arbitrary number of hexagons, though the solids then obtained do no longer belong to the family af Archimedean solids. They find there place in the Fuller domes, and have also be found in near-spherical C_n structures, in which, next to n = 60, also values of n = 70, 76, 84, 90 and 94 have been shown to exist.

3.13 THE TRUNCATED DODECAHEDRON (3 10 10)

The truncated dodecahedron is bounded by 12 10-gons and 20 triangles and has similar properties as the (5 6 6) (Figure 3.14). Its vertex coordinates are given by:

$(0\ \pm 2\sqrt{5}\ \pm(3 - 2\sqrt{5}))$ cycl, the 12 vertices in faces of a cube,

$(\pm 4\ \pm 2\ \pm(3 - \sqrt{5}))$ cycl, the 24 vertices adjacent to the first series,

$(\pm 2\ \pm(\sqrt{5}+1)\ \pm(2\sqrt{5} - 2))$ cycl, the remaining 24 vertices.

With these coordinates the edge length is $l = 2(3 - \sqrt{5})$.

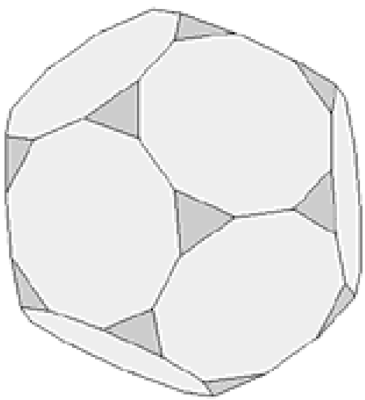


Figure 3.14 (3 10 10)

A similar series of truncations as we have seen in Figure 3.8, is being represented for the (5 6 6) and the (3 10 10) in Figure 3.15, in which an R12 is transformed into an R20 via a (3 10 10), a (3 5 3 5) and a (5 6 6).

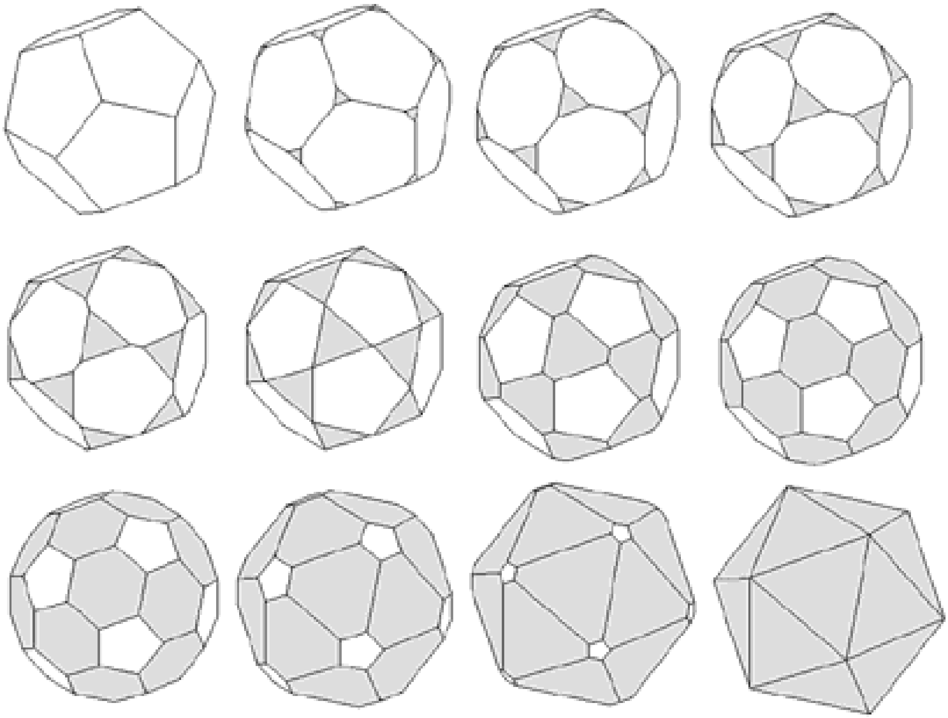


Figure 3.15 Transformation by truncations

3.14 THE RHOMBI-ICOSI-DODECAHEDRON (3 4 5 4)

Just as the (3 4 4 4) can originate from R8 or R6 by truncation along the edges, the (3 4 5 4) can be formed from the R20 or the R12. With 20 triangular faces this solids fits into an R20; with its 12 pentagons into an R12. Its 30 squares arise from "squaring" the 30 edges of R20 or R12 (Figure 3.16).

Of its 60 vertices 24 lie in the faces of a surrounding cube (six squares). Their coordinates are: (±a ±a ±(2+a)) cycl. The coordinates of the other vertices are: (0, ±(a+1) ±(2a+1)) cycl and (±1 ±2 ±(a+1)) cycl, in which $a = (\sqrt{5} - 1)/2$. The edge length of the circum-cube is 4+2a and of (3 4 5 4): 2a.

The growth of the edges of an R12 into rectangles is illustrated in Figure 3.17. When the rectangles become squares, the (3 4 5 4) is obtained; with further broadening the solid approaches to an R20.

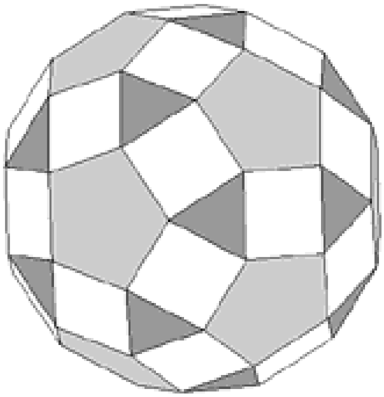


Figure 3.16 (3 4 5 4)

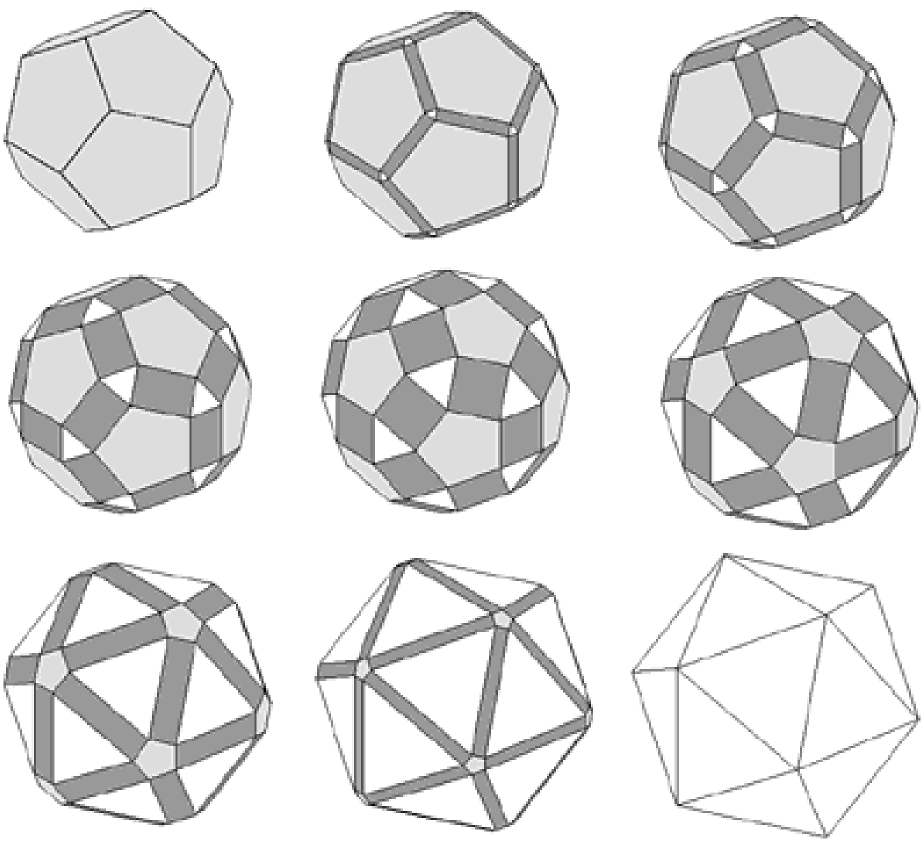


Figure 3.17 (3 4 5 4) as formed from an R12 or an R20

3.15 THE TRUNCATED ICOSIDODECAHEDRON (4 6 10)

Just as we have seen with the (4 6 8), the name “truncated icosi-dodecahedron” is not correct. After truncation the rectangular faces have to be shifted until they form squares. Then the elongated edges of the 6-gons and the 10-gons do not have common points of intersection any more (Figure 3.18).

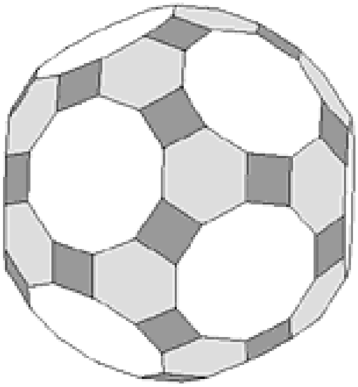


Figure 3.18 (4 6 10)

The coordinates of the 120 vertices are, in their simplest form:

$[\pm(3-\sqrt{5})\quad \pm(3-\sqrt{5})\quad \pm(3\sqrt{5}-1)]$ cycl,

$[\pm 2\quad \pm(3+\sqrt{5})\quad \pm(6-2\sqrt{5})]$ cycl,

$[\pm(\sqrt{5}+1)\quad \pm(7-\sqrt{5})\quad \pm(3-\sqrt{5})]$ cycl,

$[\pm 2\sqrt{5}\quad \pm(5-\sqrt{5})\quad \pm(2\sqrt{5}-2)]$ cycl,

$[\pm 4\quad \pm 2\quad \pm(3\sqrt{5}-3)]$ cycl.

De edge length of (4 6 10) is, with these coordinates, $l = 2(3-\sqrt{5})$.

3.16 THE SNUB CUBE (3 3 3 3 4)

The Archimedean solids described so far could all be considered as originating in a relatively simple way from truncation, once or twice, of vertices or along edges. With (3 3 3 3 4) this is not the case; this solid cannot be “discovered” as easily as the other ones. Nevertheless it is narrowly related to the cube, since it contains six faces which are within the faces of a cube (Figure 3.19). These faces have, however, undergone a certain rotation; the position of the vertices is determined by two independent conditions for equality of the edges. These two conditions lead to an equation of the third degree, which can quite easily be solved.

The coordinates of the 24 vertices can be expressed as:

$(\pm a\ \pm a^2\ \pm 1)$ cycl with $xyz < 0$,

$(\pm a\ \pm 1\ \pm a^2)$ cycl with $xyz > 0$.

a is the root of the equation: $a^3 + a^2 + a = 1$; $a = 0.54369$; $a^2 = 0.29560$. The edge length follows from $l^2 = 2(a^2 + 2a - 1)$.

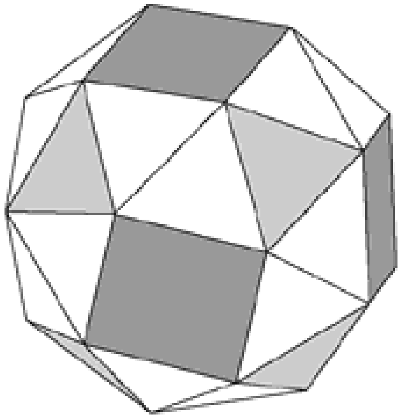


Figure 3.19 (3 3 3 3 4)

If the condition for the signs for the coordinates is exchanged, then a similar polyhedron results which is the mirror image of the first one. Thus there exist two modifications of (3 3 3 3 4): one showing a rotation to the right of the squares with respect to the cube faces, and one to the left.

Of the 32 triangular faces, 8 take a special position; these are all surrounded by other triangles. It appears that the planes of these 8 triangles are perpendicular to the diagonals of the circum-cube; when extended, these faces therefore form a regular octahedron. Within the faces of this octahedron these faces of the (3 3 3 3 4) are rotated in a similar way as the square faces within the faces of the cube. The other 24 triangular faces form, two by two, the interfaces between two square faces (and also between two triangles of the series mentioned before); this suggests an analogy with the edges of a cube (and of an octahedron).

3.17 THE SNUB DODECAHEDRON (3 3 3 3 5)

The (3 3 3 3 5) is analogous to the (3 3 3 3 4); instead of a cube we take a dodecahedron {5,3}, and we inscribe within each pentagonal face smaller regular pentagons, somewhat rotated with respect to the large pentagon (Figure 3.20). With the proper choice of the rotation angle and the reduction factor the distances between adjacent vertices are equal, and a uniform polyhedron is formed, built-up from pentagons and triangles.

As a matter of fact, (3 3 3 3 5) possesses 12 pentagonal faces, 20 triangular faces fitting into a regular icosahedron, and 60 triangular faces which, two by two, form the boundaries between the pentagons.

When we try to calculate the coordinates of the vertices, we are confronted with a fourth-grade equation with irrational coefficients, which will not be represented here. After solving this equation the coordinates appear to be as follows:

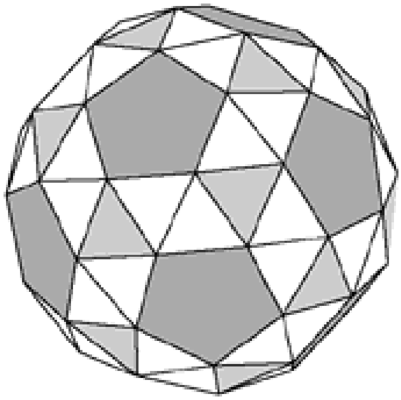


Figure 3.20 (3 3 3 3 5)

(±3.22 ±18.01 ±2.84) cycl with xyz > 0,
(±4.88 ±16.99 ±5.52) cycl with xyz < 0,
(±12.15 ±12.49 ±6.26) cycl with xyz < 0,
(±15.00 ±10.73 ±1.66) cycl with xyz > 0,
(±9.47 ±14.14 ±7.28) cycl with xyz > 0.
The edge length is, with these coordinates, $l = 8.588$.
This polyhedron has, as (3 3 3 3 4), two modifications, one with a positive rotation and one with a negative one, which are each other's mirror images.
When we consider (3 3 3 3 4) and (3 3 3 3 5) as a series of related solids, then it is intriguing to try to extend this series to both sides. (3 3 3 3 6) is, as a matter of fact, a plane tessellation, which has the same structure as both polyhedra (see 3.19).
(3 3 3 3 3) is the regular icosahedron {3,5}, which, with rotated faces, fits into the R4 as (3 3 3 3 4) into R6 and in R8, and (3 3 3 3 5) into R12 and in R20.

3.18 SURFACE AREAS AND VOLUMES

Surface areas and volumes of the various Platonic and Archimedean solids are interesting because of the question: In how far do these solids approach to the sphere, in other words: how compact are they? From superficial inspection we see that the regular tetrahedron forms a negative extreme (not counting the higher prisms and antiprisms). Within the family of regular and uniform solids it seems that the sphere is the better approached as the number of faces increases.
This problem is of importance when a dome has to be constructed or an as spherical as possible football (with the latter, of course, not considering the effect of inflation)
To investigate the “compactness” nearer, we can use various criteria:

- We can compare the solid with its circum-sphere, and calculate its volume compared with that of this sphere.
- Compared with this same sphere, their areas can be compared.

- A measure for the compactness could be the angle at which we see, from the midpoint, an edge.
- The "index of compactness" (see later), which is the ratio between the area of the solid to that of a sphere with the same volume.
- Finally we can look at the ratio between the smallest and the biggest distance from the centre to vertices or faces of the solid.

We shall consider each of these criteria, whereby the first question is: how big are the surface areas and the volumes?
The areas can be easily calculated. The area of a regular n-gon, {n}, is given by:

$$A_n = (nl^2/4)\cdot\text{cotg}(180°/n),$$

thus the total area of, e.g. (4 6 10), consisting of 30{4}, 20{6} and 12{10}, is:
$(30/4)l^2\cdot\text{cotg } 45° + (20/4)l^2\cdot\text{cotg } 30° + (12/4)l^2\cdot\text{cotg } 18° = 30\cdot l^2(1+\sqrt{3}+\sqrt{(5+2\sqrt{5})})$, ($l$ is the edge length).
The volume is the sum of a number of pyramids, all with their top in the centre. The height of a pyramid is the distance of the centre to the appropriate face, and can easily be calculated from the coordinates of the vertices, e.g. as follows: For one of the 10-gons of (4 6 10) two opposing corners lie at $(-2, 3+\sqrt{5}, 6-2\sqrt{5})$ and $(2, 3\sqrt{5}-3, 4)$; the middle of this face is then: $(0, 2\sqrt{5}, 5-\sqrt{5})$ and the distance p_{10} is given by: $p_{10}^2 = 0^2 + (2\sqrt{5})^2 + (5-\sqrt{5})^2 = 50 - 10\sqrt{5}$.
The table presents some geometrical data on the Platonic and the Archimedean solids, inclusive a number of prisms and antiprisms. The volumes V and the areas A are expressed in those of the circum-spheres ($V/(4\pi r^3/3)$ en $A/4\pi r^2$), while r/l is a (reciprocal) measure for the angle φ at which the centre "sees" an edge; this angle is given by:

$$\cos\varphi = 1 - l^2/2r^2.$$

An interesting measure for the compactness of a solid is also the "index of compactness", ci, introduced by Drs A. Verweij ("Meetkunde voor Bouwkunde", TU Delft diktaat a19, August 1991) as the ratio between the area of the solid to that of a sphere with the same volume; this sphere has a radius $r_b = (V\cdot 3/4\pi)^{1/3}$, so that $ci = (36\pi V^2)^{1/3}/A$. The big advantage of this parameter is that it can be applied to other types of solids, such as Archimedean solids of the second kind, which do not possess a circum-sphere, but, in particular, to arbitrary spatial constructions such as buildings. For the solids considered in this chapter the index of compactness is related to other parameters by:

$$ci = (V/V_b)^{2/3}/(A/A_b),$$

and has also been tabulated.

Finally the table gives a direct measure for the "unroundness", namely the ratio, λ, between the maximum and the minimum distance from the centre to any point of the polyhedron. With (4 6 8), e.g., the distance to a vertex (r) is 1.0000, to a square: 0.9523, to a hexagon: 0.9021, and to an octagon: 0.8259. The biggest deviation from a sphere is thus found in the centres of the octagons, so that λ = 1/0.8259 = 1,2107.

polyhedron	V/V_S	A/A_S	r/l	φ	H/m	ci	λ
(3 3 3)	.123	.368	0.61	109.5	1.33	.671	3
(3 3 3 3)	.318	.551	0.71	90	1.5	.846	1.732
(4 4 4)	.368	.637	0.87	70.5	2.67	.806	1.732
(3 3 3 3 3)	.605	.762	0.95	63.4	2.4	.939	1.258
(5 5 5)	.665	.837	1.40	41.8	6.67	.910	1.258
(3 4 3 4)	.563	.753	1	60	3	.905	1.414
(3 6 6)	.401	.702	1.17	50.5	4	.775	1.915
(4 6 6)	.683	.853	1.58	36.9	8	.910	1.291
(3 8 8)	.577	.816	1.77	32.7	8	.849	1.474
(3 4 4 4)	.760	.873	1.40	41.9	6	.954	1.159
(4 6 8)	.802	.915	2.32	24.9	16	.943	1.211
(3 5 3 5)	.780	.891	1.62	36	7.5	.951	1.176
(5 6 6)	.867	.941	2.48	23.3	20	.967	1.093
(3 10 10)	.776	.911	2.97	19.4	20	.926	1.193
(3 4 5 4)	.892	.947	2.23	25.9	15	.979	1.082
(4 6 10)	.898	.959	3.80	15.1	40	.970	1.036
(3 3 3 3 4)	.776	.875	1.34	43.7	4.8	.965	1.176
(3 3 3 3 5)	.896	.947	2.16	26.8	12	.982	1.088
(4 4 3)	.232	.527	0.76	81.8	2	.716	2.646
(4 4 5)	.427	.690	0.99	60.9	3.33	.823	1.973
(4 4 6)	.444	.713	1.12	53.1	4	.816	2.236
(4 4 8)	.421	.718	1,40	41,9	5,33	,782	2,798
(4 4 10)	.378	,704	1,69	34,3	6,67	,742	3,387
(4 4 12)	.336	,687	2.00	29.0	8	.704	3.991
(3 3 3 4)	.410	.642	0.82	74.9	2	.859	1.957
(3 3 3 5)	.438	.684	0.95	63.4	2.5	,844	2.236
(3 3 3 6)	.434	.699	1.09	54.7	3	.820	2.542
(3 3 3 8)	.391	.698	1.38	42.6	4	.767	3.198

From the table it appears that (4 6 10) excels in several respects: it is, by far, the biggest polyhedron for a given edge length. It fills the circum-sphere with the highest percentage, namely 89.8 %, followed by (3 3 3 3 5) and (3 4 5 4) with, respectively, 89.6 and 89.2 %. The ratio's of the areas show the same order of sequence, viz. 95.9 , 94.7 and 94.7 %.
Also the “unroundness” of (4 6 10) has the lowest value, namely 1.036. The index of compactness shows a different picture: it is highest for (3 3 3 3 5), followed by (3 4 5 4) and (4 6 10).
The footballs (5 6 6) and (3 10 10), do not score highest in any respect!

It is interesting to investigate in how far the various geometrical constants are related to each other. It appears that between e.g. V/V_b and r/l only a very global relation exists with much scatter. As a matter of fact the prisms (4 4 n) and the antiprisms (3 3 3 n) deviate the strongest; for these V/V_b approaches to zero with increasing n and A/A_b to 0.5 ($2\pi r^2/4\pi r^2$).
Other trials to discover a correlation between the various quantities result in an unexpected relation between r/l and the number of vertices, divided by their functionality, V/m. For a cube, for instance, V/m = 8/3, for (3 3 3 3 5), 60/5 = 12 etc. This correlation is demonstrated in Figure 3.21; it appears that even the members of the prism- and antiprism series fit into this pattern, which, on average, can be expressed as:

$$r/l = 0.6 - \sqrt{(V/m)}.$$

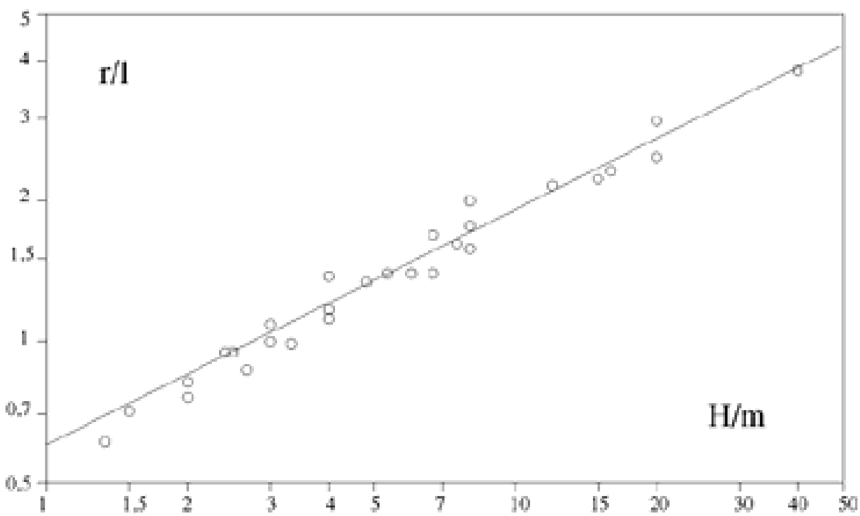


Figure 3.21 Relation between r/l and V/m

3.19 SERIES

When we discussed at the first time the uniform polyhedra, we already met two clearly recognizable series, namely the series (4 4 n) and (3 3 3 n). Next to these, also other series exist, such as (3 3 3), (3 4 4), (3 6 6), (3 8 8), (3 10 10), (3 12 12). This series starts off with the regular tetrahedron, then the smallest Archimedean prism follows, then three truncated Platonic solids, while the series ends with (3 12 12), which is not a polyhedron but a plane tessellation.

A survey of the various possibilities to obtain a systematic classification into series, is given in Figure 3.22. The starting point of all series is the tetrahedron (3 3 3), from which the series (3 3 3), (3 3 3 3), (3 3 3 3 3), (3 3 3 3 3 3), and also the series (3 3 3), (4 4 4), (5 5 5), (6 6 6), and the series mentioned above originate.

The other series form branches of these three series. Some of the series cross-over in a remarkable way, such as (4 4 4), (5 5 5), (6 6 6) with (4 4 5), (5 5 5), (5 6 6).

When we also consider shorter series we meet in this schedule all eleven possibilities for plane tessellations with regular polygons, namely (3 3 3 3 3 3), (4 4 4 4), (6 6 6), (3 3 3 3 6), (3 4 6 4), (3 6 3 6), (3 12 12), (3 3 3 4 4), (3 3 4 3 4), (4 6 12) and (4 8 8). Three of these eleven are regular and eight of them are semiregular or uniform. A remarkable pair is formed by (3 3 3 4 4) en (3 3 4 3 4), which comprise the same elements; while the first is rather trivial and related to the prisms and antiprisms, the second one shows a surprising pattern. Figure 3.23 gives a survey of the tessellations mentioned.

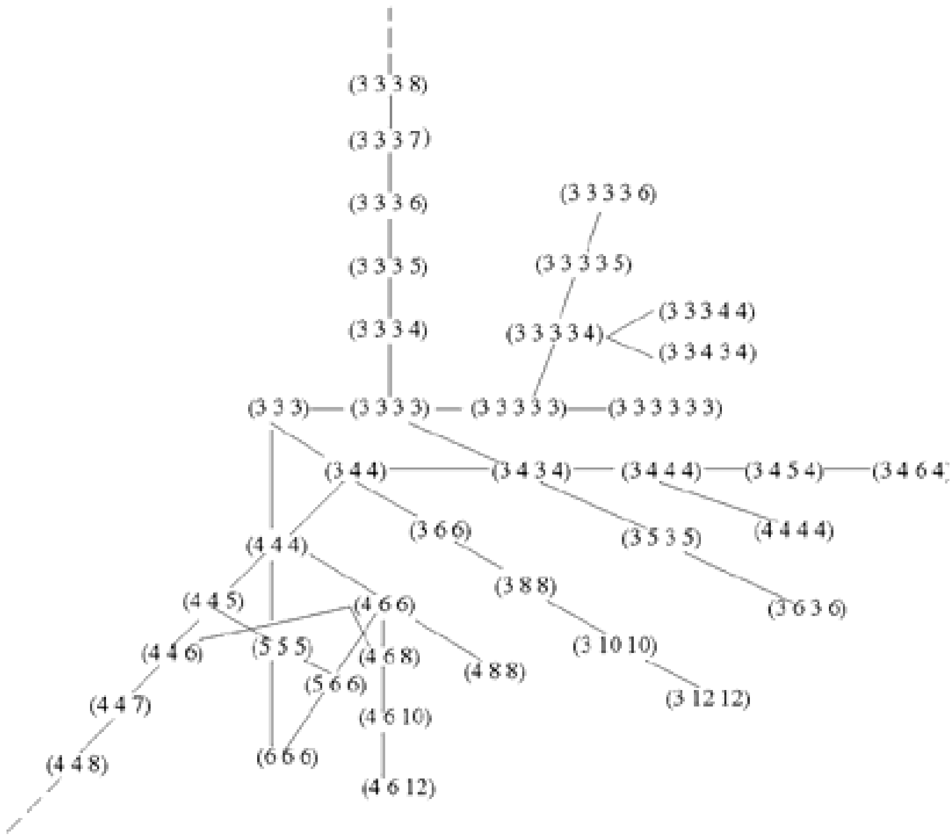


Figure 3.22 Series of polyhedra, including plane tesselations

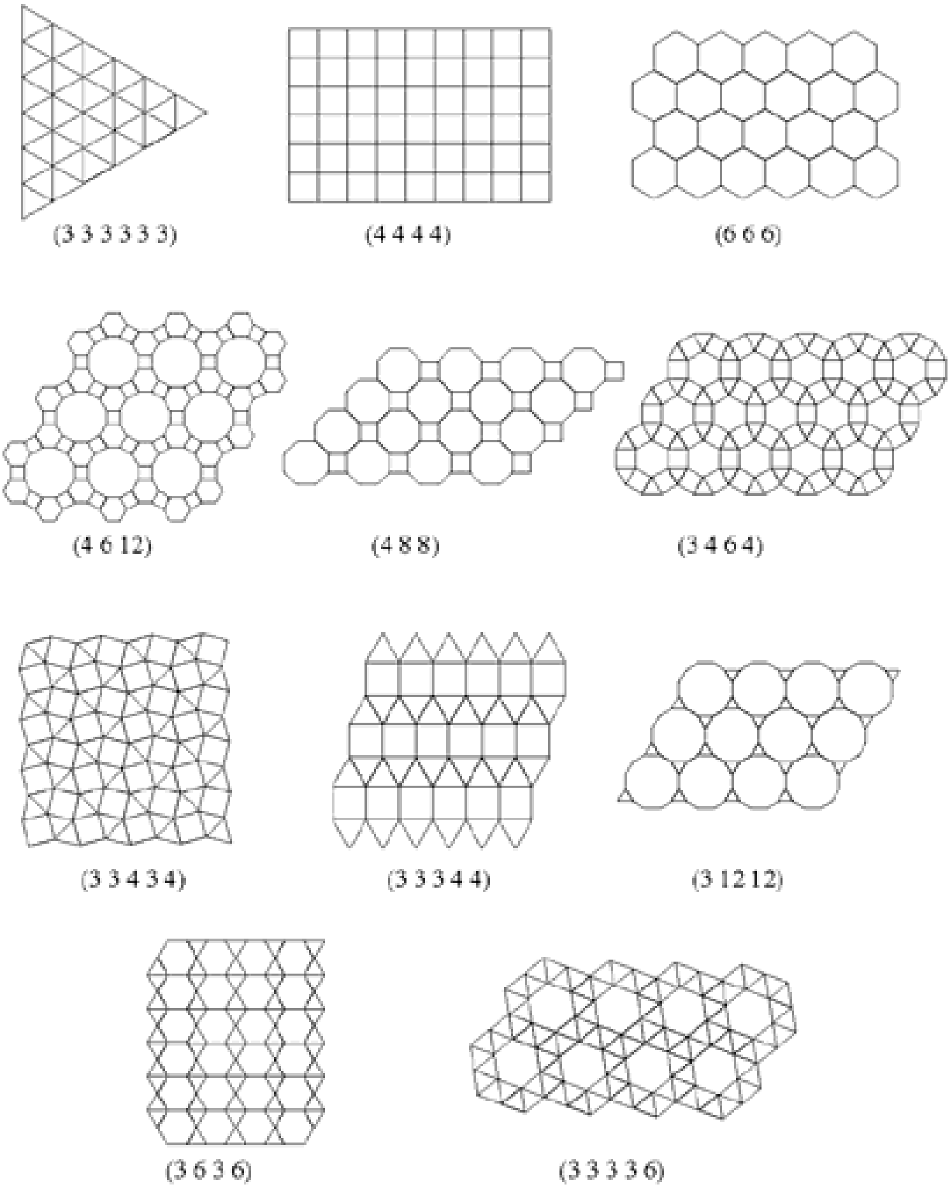


Figure 3.23 Three regular and eight uniform plane tessellations

4

SEMIREGULARITY INVERTED (UNIFORM POLYHEDRA OF THE SECOND KIND)

4.1 INTRODUCTION

In Chapter 3 the difference between uniform polyhedra of the first and the second kind has already been mentioned. From each type of the first kind (U1) an analogous type of the second kind , its dual, (U2) can be derived by interchanging faces and vertices in their definition. Just as the faces of a U1 are regular but unequal, the vertices of a U2 form regular polyhedral angles, but there are at least two different types. While with a U1 all vertices are equal but not regular, a U2 is bounded by equal but irregular polygons.

In order to represent the uniform polyhedra of the second kind by a simple notation, we use the same notation as with the first kind, but now with double brackets instead of single ones. The dual of (3 4 3 4) is, therefore, indicated as ((3 4 3 4)) etc. Apparently for the Platonic solids two different notations are valid, e.g. the cube can be indicated by (4 4 4) as well as by ((3 3 3 3)), the tetrahedron by (3 3 3) or by ((3 3 3)).

For the description of the various U2's we can follow two ways: The shape of the faces can be determined rather simply from the condition that the polyhedral angles are regular, while all dihedral angles between the faces are equal. The faces and the vertices can be found by either connecting the midpoints of the faces of the U1, or by applying tangent planes to the circum-sphere round the U1 at its vertices.

4.2 CALCULATION OF THE SHAPE OF THE FACES

Since the polyhedral angles at the vertices are regular, the general equation, given in 1.5 , for the dihedral angles φ between the faces of a regular m-hedral angle is valid:

$$\cos \varphi = \frac{\cos \alpha - p}{\cos \alpha + 1}$$

in which a is the angle of the polygon at the corner which, with m–1 other ones, forms the m-hedral angle, while

$$p = 1 + 2\cos(360°/m).$$

When a polyhedron contains m_1-hedral angles and m_2-hedral angles (derived from a U1 bounded by m_1-gons and m_2-gons), then:

$$\cos\varphi = \frac{\cos\alpha_1 - p_1}{\cos\alpha_1 + 1} = \frac{\cos\alpha_2 - p_2}{\cos\alpha_2 + 1}$$

since for both polyhedral angles the dihedral angles φ are equal. p_1 and p_2 are known from m_1 and m_2; to solve the equation we need a second relation between α_1 and α_2. This is provided by considering the sum of the angles in a face: if the face is triangular with an angle α_1 and two angles α_2, then $\alpha_1 + 2\cdot\alpha_2 = 180°$ or $\cos\alpha_1 = -\cos(2\cdot\alpha_2)$.
In general the above formula for φ is valid for n-gons with angles $\alpha_1, \alpha_2, ..., \alpha_n$ with, respectively, m_1-, m_2-,...m_n- hedral angles; the second relation is then: $\sum\alpha_i = (n-2)\cdot 180°$.
This is a set of n+1 equations in n+1 variables, namely $\alpha_1 .... \alpha_n$ and φ. The solution can easily be found when the faces are isosceles triangles, such as with ((4 6 6)), ((3 8 8)) etc., and also for the series derived from prisms ((4 4 m)). Much more effort is needed for scalene triangles, ((4 6 10)) and n-gons with n > 3 such as ((3 3 3 3 5)). In these cases is is sometimes more efficient to change the strategy and to choose the much more general approach of calculating the coordinates of the vertices of a U2 from the equations of the tangent planes to the sphere round U1.
When the angles are known, the length ratio of the sides is, except for a triangle, not automatically fixed; this can, however, be derived from the evident condition that edges between corresponding pairs of vertices should have equal length.
Further on in this chapter the face shapes of the various U2's will be considered in more detail.

4.3 CALCULATION OF VERTEX COORDINATES AND EQUATIONS OF PLANES

As already remarked in § 4.1, the coordinates of a U2 can be determined as the midpoints of faces of the corresponding U1 or from the tangent planes to the circum-sphere. Though the first way seems simpler, we choose the second method, since this is more elegant and straight-forward.
We apply at the vertex $P_1(x_1,y_1,z_1)$ of the U1 the tangent plane, which is the plane through P_1 perpendicular to the line connecting P_1 to the centre of the polyhedron. The length of this line segment (the radius of the circum-sphere), is:

$$R = (x_1{}^2 + y_1{}^2 + z_1{}^2)^{1/2} .$$

The equation of the tangent plane is:

$$x_1(x - x_1) + y_1(y - y_1) + z_1(z - z_1) = 0$$

or $$x_1x + y_1y + z_1z - R^2 = 0$$

For two other corners of one of the polygons meeting at P_1, viz. P_2 and P_3, the equations of the tangent planes are similar, but now with indices 2 and 3, respectively. A vertex of the U2 is the point of intersection Q of these three planes; its coordinates can, therefore, be found by solving the three equations. For this purpose we use the notation with determinants:

$$x = \frac{R^2}{\Delta}\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix} \quad y = \frac{R^2}{\Delta}\begin{vmatrix} x_1 & 1 & z_1 \\ x_2 & 1 & z_2 \\ x3 & 1 & z_3 \end{vmatrix} \quad z = \frac{R^2}{\Delta}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x3 & y_3 & 1 \end{vmatrix} \quad \Delta = R^2\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x3 & y_3 & z_3 \end{vmatrix}$$

in which $R^2 = x_1{}^2 + y_1{}^2 + z_1{}^2 = x_2{}^2 + y_2{}^2 + z_2{}^2 = x_3{}^2 + y_3{}^2 + z_3{}^2$.
The determinant, e.g. for Δ, is an abbreviated notation for:

$$x_1(y_2 \cdot z_3 - y_3 \cdot z_2) - x_2(y_1 \cdot z_3 - y_3 \cdot z_1) + x{\pm}_3(y_1 \cdot z_2 - y_2 \cdot z_1).$$

In the equations for all three coordinates, R^2 appears as a multiplication factor; when we let R = 1, the calculations are simplified, while the mutual relations between the coordinates are not affected.
Independent of the type of polygon in the U1, the calculation of a vertex of the U2 only requires three tangent planes. Though the approach with determinants requires some rather complicated calculations, it is still a surprisingly simple method to characterize the U2's, (in particular when using a computer!). By calculating the distances between adjacent vertices we obtain an independent check on both methods to determine the shape of the faces.

4.4 ARCHIMEDEAN DOUBLE-PYRAMIDS

The double-pyramids are the reciprocals of the prisms; they are formed by two identical pyramids with a common base.
The simplest one is the ((3 4 4)), the faces of which are isosceles triangles with a top-angle of about 97° (arccos (–1/8)). It is bounded by six of these triangles, which form two three-sided pyramids with an equilateral triangle as common base.
The next double-pyramid is the ((4 4 4)), which is nothing else than the (3 3 3 3) or {3,4}, the regular octahedron. The cube appeared also to be the second in the series of the Archimedean prisms! The top-angle of the triangles is, of course, 60°.

Then ((4 4 5)) and ((4 4 6)) etc. follow, bounded by 10, 12 etc.triangles with top-angles of 40.4° (arccos(5(√5+1)/16), 29.0° (arccos 0.875) etc.
The general formula for the top-angle α_m of the faces of a ((4 4 m)) is:

$$\cos \alpha_m = \cos \theta + \frac{1}{2} \sin^2 \theta \qquad \text{with } \theta = 360°/m \text{ ;}$$

this relation follows in a simple way from the foregoing paragraph.
It is clear that with increasing m the top-angles become smaller and smaller and approach to zero. While the Archimedean prism approaches to a flat disk, the double-pyramid becomes a two-pointed needle.
The coordinates of ((4 4 m)) can, in the most simple way, be expressed by locating the vertices on the z-axis, and the common base, {m}, in the x-y plane. Calculation of the altitudes, h, of the pyramids is, easier than with the determinant method, performed by using the following relation:

$$h/l = \frac{\cos(\theta/2)}{1 - \cos \theta}$$

in which l is the edge length of {m} and $\theta = 360°/m$.
For m = 3, 4, 5, 6 and 8 we find, respectively:

$$h/l = 1/3 \text{ , } \sqrt{2}/2 \text{ , } (5+3\sqrt{5})/10 \text{ , } \sqrt{3} \text{ and } \sqrt{(5 + 7\sqrt{2}/2)}.$$

Fig 4.1 gives a survey of a number of double-pyramids.

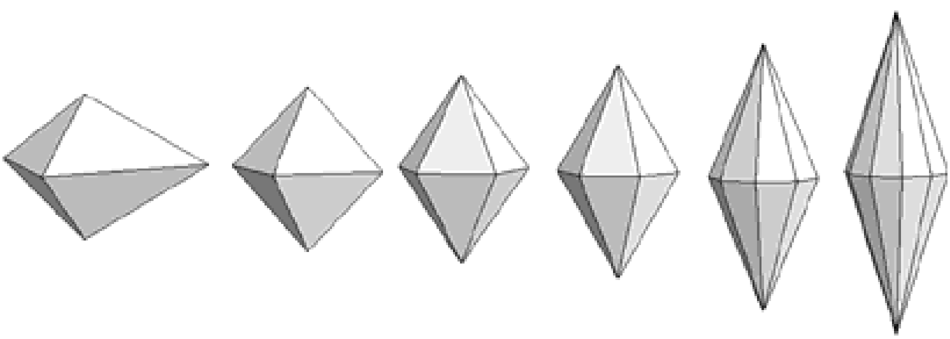


Figure 4.1 Some Archimedean double-pyramids

4.5 ARCHIMEDEAN TRAPEZOEDERS

The trapezoeders originate from the antiprisms by reciprocation. The ((3 3 3 m)) is bounded by 2m tetragons. Just as with the ((4 4 m)) there are two tops, forming m-hedral angles; the other vertices form trihedral angles and are, in a zig-zag shape, grouped around a plane perpendicular to the axis. The faces are deltoids; from the condition of equal dihedral angles a relation follows for the angles of these tetragons (see Figure 4.2):

$$\cos \alpha_3 = (1 - \sqrt{q})/2 ,$$

$$\cos \alpha_m = (3q - q\sqrt{q} - 2)/2 ,$$

in which q = 2 + 2cos(360°/m).

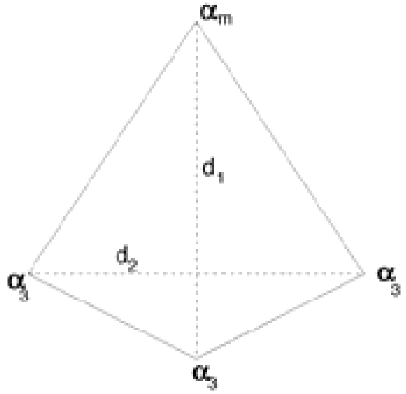


Figure 4.2 Face of a trapezohedron

The ratio of the diagonals is given by:

$$d_1^2/d_2^2 = (1 + \cos \alpha_3) / (1 - \cos \alpha_m)$$

For some of the trapezoeders the angles (in degrees) and the ratio of their diagonals are given in the table below:

m	α_3	α_m	d_1/d_2
3	90	90	1
4	102,0	54,1	1,38
5	108	36	1,90
6	111,5	25,6	2,54
8	115,1	14,8	4,17
10	116,8	9,6	6,28

The first of this series, ((3333)), is of course the cube {4,3}. As its "tops" we can define two opposite vertices; its axis is the diagonal connecting these tops. Further on in this series we see that the top angle of the faces gradually decreases to zero with increasing m; the other three angles approach to 120°.
Figure 4.3 presents pictures of the species mentioned in the table.

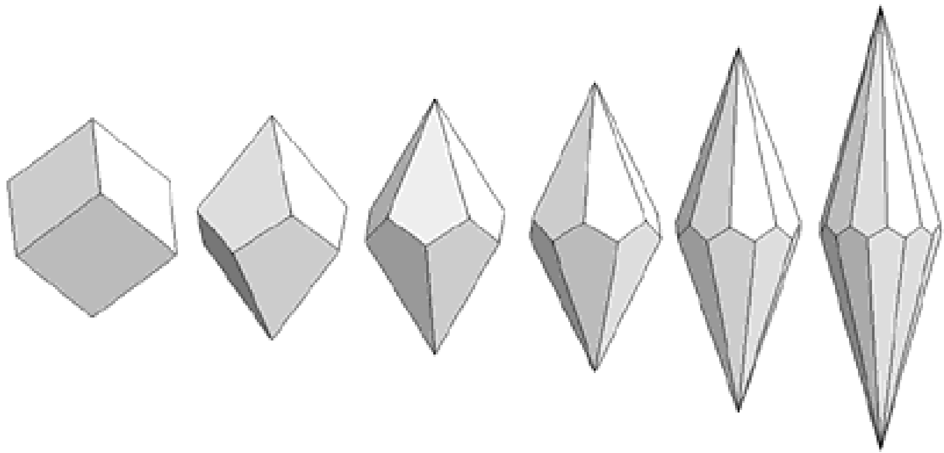


Figure 4.3 Archimedean trapezoeders

4.6 THE RHOMB-DODECAHEDRON

The ((3 4 3 4)) is called the rhomb-dodecahedron; it is derived from the (3 4 3 4) by dual interchange, and thus it possesses 12 faces and 14 vertices. Of these vertices 8 form a trihedral angle and 6 a tetrahedral angle (the (3 4 3 4) is bounded by 8 triangles and 6 squares). The faces are rhomb-shaped (equal edges), which explains the name of the solid. Because of the equality of the edges this solid takes a special place in the series of U2's. This is in analogy to the special character of the (3 4 3 4), which shows equal dihedral angles along all its edges (with edges, length and dihedral angles are dually related quantities). The same situation is met with (3 5 3 5) and ((3 5 3 5)).
The angles of the rhombs, α and β, are given by: $\cos\alpha = -\cos\beta = 1/3$; $\alpha = 70.53°$ and $\beta = 109.47°$. The mutually perpendicular diagonals have a ratio of $1 : \sqrt{2}$.
In the same way as the (3 4 3 4) is narrowly related to the cube, {4,3}, and the octahedron, {4,8}, this is the case with the ((3 4 3 4)). Its long plane diagonals are the 12 edges of a regular octahedron, its short diagonals those of a cube, so that six of the vertices form the vertices of an R8, and the remaining eight those of an R6. This is demonstrated in Figure 4.4, which is comparable to Figure 2.9.
When we calculate the dihedral angle between pairs of adjacent faces, this appears to be, for all pairs, 120°. This exceptional value (360°/3) indicates that three of these solids can be joined together seamlessly. From further analysis it follows that we can go on in all directions; this solid is able to fill space, in the same way as the cube, some prisms, and the (4 6 6), the truncated octahedron.

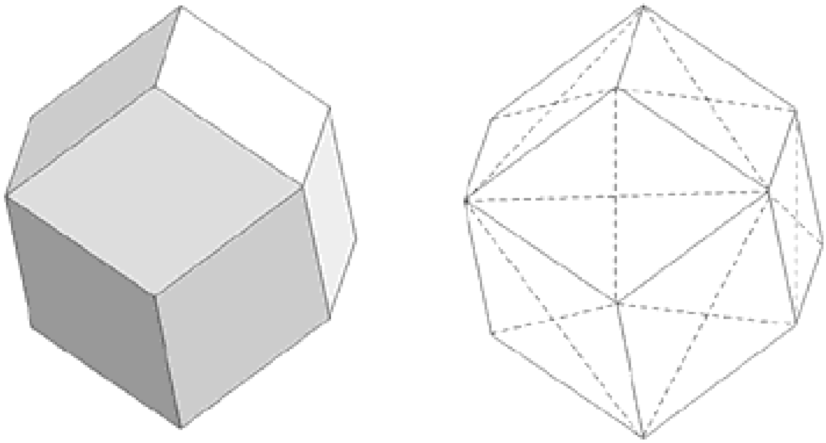


Figure 4.4 The rhomb-dodecahedron ((3 4 3 4))

This fact implies a very interesting consequence. The rhomb-dodecahedron can be split-up into 24 tetrahedra by dividing each of the twelve rhombs along its short diagonal into two triangles; each of these triangles forms, together with the midpoint of the solid, a non-regular tetrahedron (see Figure 4.5). Independent of the initial ((3 4 3 4)), this tetrahedron is able to fill space in all directions. This is interesting enough to devote, as an intermezzo, a special paragraph to it.

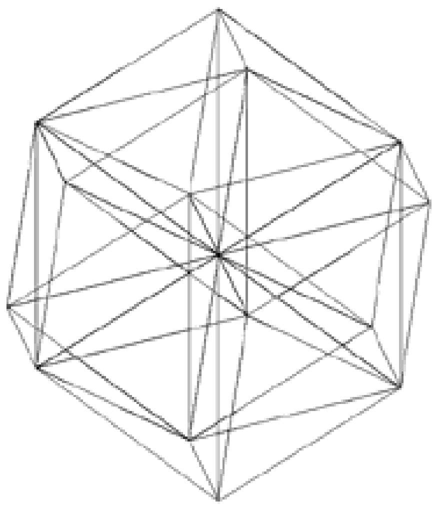


Figure 4.5 The rhomb-dodecahedron, split-up into 24 tetrahedra

4.7 INTERMEZZO : SPACE-FILLING TETRAHEDRA

In Chapter 2 we have already seen that space cannot be filled with regular tetrahedra; we need additional regular octahedra. The question is now: does a tetrahedron exist

which is able to fill space; as a matter of fact a non-regular one, but with equal faces, in other words: its four faces should be congruent and also its four trihedral angles. Such a tetrahedron can be formed from any scalene triangle (see Figure 1.4).
Obviously only tetrahedra composed of isosceles triangles have to be considered; with scalene triangles it would never be possible to fit two faces together. Consequently, there are only two kinds of edges and two kinds of dihedral angles. When we denote the angles of the triangle by α_1, α_1 and α_2 and the dihedral angles by a_1, a_1 and a_2, then the latter can, with the aid of form. 1.1 be expressed as:

$$\cos a_1 = \frac{\cos\alpha_1 - \cos\alpha_1 \cdot \cos\alpha_2}{\sin\alpha_1 \cdot \sin\alpha_2} \quad ; \cos a_2 = \frac{\cos\alpha_2 - \cos^2\alpha_1}{\sin^2\alpha_1}$$

With $\alpha_2 = 180° - 2\alpha_1$ and the well-known formula's for double angles, the α's in this formula can be eliminated, which results in a relation between a_1 and a_2:

$$2\cos a_1 + \cos a_2 = 1 \quad .$$

Now we have to require, as a condition for space-filling, that a_1 as well as a_2 are divisors of 360°. After simply trying with $a_1 = 360°/n_1$, it appears that only $n_1 = 6$ and $n_2 = 4$ meet this requirement, so that the only solution is: $a_1 = 60°$, $a_2 = 90°$. The angles α_1 and α_2 are then, respectively: 54.7° and 70.5°.
The tetrahedron found in this way can be easiest represented with x, y, z coordinates : (–2, 1, 0), (2, 1, 0), (0, –1, 2), (0, –1, –2). It can also be originated from a prismatic bar with an equilateral cross-section (edge length = a) by putting this bar with one of its sides on a horizontal plane, cutting it by a plane perpendicular to this plane but at an angle of 35.26° (arctan $1/\sqrt{2}$) with the plane perpendicular to the axis of the bar, then rotating the bar by 120° and shifting it over a distance of 0.3536·a, cutting it again, a.s.o. (Figure 4.6).

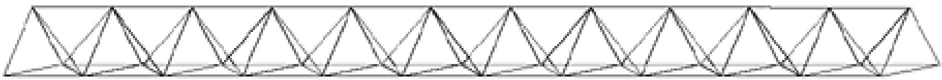


Figure 4.6 Bar of space-filling tetrahedra

The space-filling nature of the tetrahedron is clearly demonstrated by this way of origination. Since the tetrahedron is equifacial, at each of its faces a bar, composed of identical tetrahedrons, can be joined. This would not be possible when the tetrahedra would be cut from the triangular bar at another angle; these tetrahedra would be able to fill space only in the direction of the bar, allowing the bars to be assembled in a parallel way only.
The space-filling tetrahedron found this way is exactly the same as the one we found in the preceding paragraph by splitting-up the rhomb-dodecahedron into 24 tetrahedra.

Calculation of the spatial angle of such a tetrahedron indeed results in 1/24 of the area of a sphere.
The possibility to construct spatial structures with this tetrahedron is demonstrated in Figure 4.7.

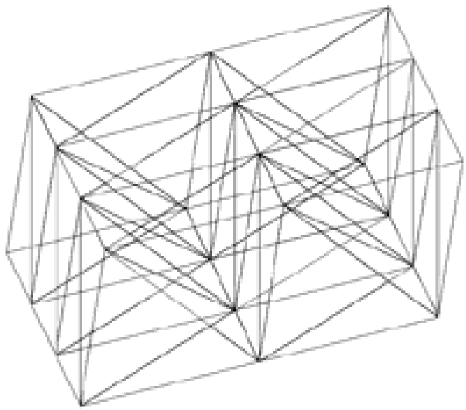


Figure 4.7 Stacking of space-filling tetrahedra

4.8 BETWEEN CUBE AND OCTAHEDRON

In Chapter 3 we saw that the cube and the octahedron can be transformed into each other in different ways, whereby some uniform polyhedra of the first kind form intermediates, such as (4 6 6), (3 8 8), (3 4 3 4) and, in another series, (3 4 4 4). A similar situation is met with the U2's, the second-kind uniform polyhedra.
The simplest method of transformation of a U1 is by truncation at the corners; then, for instance, a trihedral angle is transformed into a triangle; a single vertex is replaced by three new vertices also forming trihedral angles.
In dual analogy it is obvious to think with U2's of extension of a triangular face into three new, again triangular, faces; in other words: we raise an obtuse three-faced pyramid on the old face.
As a first illustration of this principle we consider the ((3 6 6)); here obtuse triangular pyramids have been placed on each of the four faces of a tetrahedron in such a way that the dihedral angles are equal. The solid thus has 12 faces in four groups of 3, which explains its name “triakistetrahedron” (3 * 4). Of its eight vertices four form a trihedral angle and four a hexahedral angle; these groups of four coincide with the vertices of two regular tetrahedra, whose edges are perpendicular to each other and have a length ratio of 3 : 5. Of the 18 edges, 6 have thus the relative length 3, and 12 the relative length 5. The faces are isosceles triangles with an obtuse top-angle of 112.89° (arccos –7/18) and base angles of 33.56° (arccos 5/6). Figure 4.8 shows the ((3 6 6)).

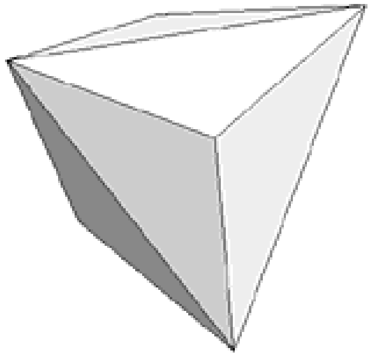


Figure 4.8 The triakistetrahedron ((3 6 6))

The U2's dealt with in the following, are more relevant to the title of this section, to start off with the ((4 6 6)). Its analogue, the (4 6 6), was obtained by truncating an R8 at its corners. The ((4 6 6)) can, therefore, be formed by adding pyramids to each face of an R6, a cube (dually related to the R8). It is bounded by $4 \cdot 6 = 24$ triangles, and is, therefore, called "tetrakishexahedron". It has 36 edges. The angles α, α and β of the triangles are given by: $\cos \alpha = 2/3$, $\cos \beta = 1/9$ ($\alpha = 48.19°$, $\beta = 83.62°$); the ratio of their sides is 3 : 3 : 4. Eight of the 14 vertices form hexahedral angles and coincide with the vertices of a cube; the other six (tetrahedral angles) form a regular octahedron. Analogous to the truncation of a cube, we can provide its dual, the octahedron, with pyramids upon its triangular faces, which results in the ((3 8 8)). This solid is also bounded by 24 faces ($3 \cdot 8$, it is thus called "triakisoctahedron") and has 36 edges. Its 14 vertices again coincide with those of a cube and an octahedron. The ratio of the edges is: 2 : 2 : $(2+\sqrt{2})$; the angles α, α and β are given by: $\cos\alpha = (2+\sqrt{2})/4$, $\cos\beta = -(2\sqrt{2}-1)/4$ ($\alpha = 31.40°$, $\beta = 117.20°$).
The number of vertices of the ((3 4 3 4)) also appeared to be 14 (see § 4.7), namely 8 of a cube and 6 of an R8. Therefore, we can imagine this solid as originating from a ((4 6 6)), by increasing the height of the pyramids till two faces which share an edge of the cube, are coplanar. These two faces than form a rhomb-shaped single face of the ((3 4 3 4)). In a similar way, raising the pyramids of the ((3 8 8)) results in a ((3 4 3 4)). The rhomb-dodecahedron can, therefore, be considered as an intermediate between the ((4 6 6)) and the ((3 8 8)). The series is shown in Figure 4.9.
Two other U2's are also related to cube, octahedron and rhomb-dodecahedron: the ((3 4 4 4)) and the ((4 6 8)). The first of these, the "deltoid-icositetrahedron" (Figure 4.10), is bounded by 24 deltoid-shaped 4-gons, the angles α, α, α and β of which are given by: $\cos \alpha = (2 - \sqrt{2})/4$, $\cos \beta = -(2+\sqrt{2})/8$; ($\alpha = 81.58°$, $\beta = 115.26°$). It has 48 edges. Eight of the 26 vertices form a trihedral angle and coincide with the vertices of a cube. Of the 18 vertices forming a tetrahedral angle, 6 coincide with the vertices of an R8 and 12 with those of a cubooctahedron (3 4 3 4), or, which is the same, with the midpoints of the faces of a cube (or an octahedron).

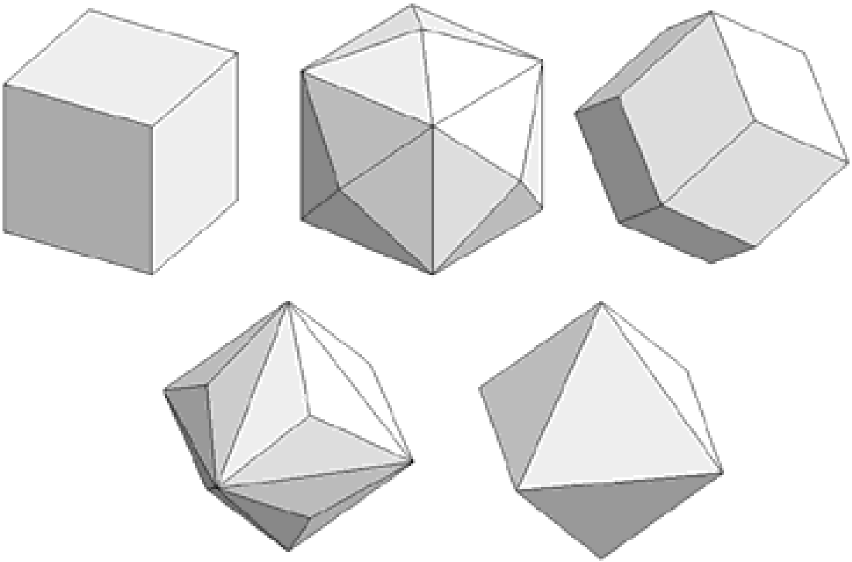


Fig.4.9 The series: cube ((3 3 3 3 3)), tetrakishexahedron ((4 6 6)) rhomb-dodecahedron ((3 4 3 4)), triakisoctahedron ((3 8 8)), octahedron ((4 4 4)).

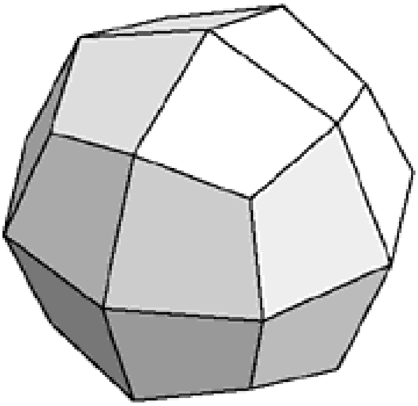


Figure 4.10 The deltoid-icosi-tetrahedron ((3 4 4 4))

The ((3 4 4 4)) can also be considered as a transition between cube and octahedron; by systematically increasing or decreasing its dihedral angles, a series of deltoid-icositetrahedrons is generated, which is found in crystal structures, and which has its extremes in the R6 and the R8.

The ((4 6 8)), the "hexakiisoctahedron" (see Figure 4.11) has 48 triangular faces, 72 edges and 26 vertices, 12 of which are tetrahedral, 8 hexahedral and 6 octahedral. The vertices show the same relation to those of the R6, the R8 and the (3 4 3 4) as with the ((3 4 3 4)).

Finally, under the heading "between cube and octahedron", the ((3 3 3 3 4)) can be mentioned, since the snub cube (3 3 3 3 4), (3.16,) could be derived from a cube as well as from an octahedron. The ((3 3 3 3 4)) is called "pentagonicositetrahedron", which simply means that it is bounded by 24 pentagons. The solid has 60 edges and 38 vertices, 32 of which are trihedral and 6 tetrahedral (Figure 4.12). Four of the five angles in a face are equal, namely 114.81°, the fifth one is 80.75°. The ratio of the edges is 1 : 1 : 1 : 1.42 : 1.42.
Just as the (3 3 3 3 4) this solid can take two different shapes which are each other's mirror image.

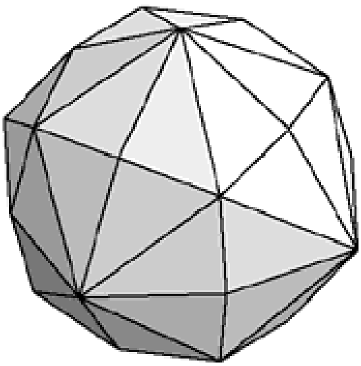


Figure 4.11 The hexakisoctahedron ((4 6 8))

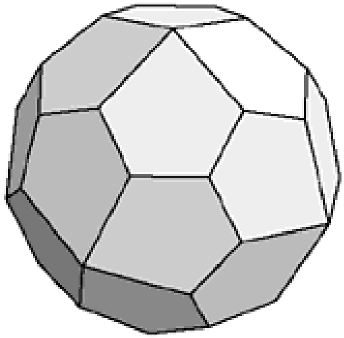


Figure 4.12 The pentagonicositetrahedron ((3 3 3 3 4))

4.9 BETWEEN DODECAHEDRON AND ICOSAHEDRON

Under this heading obviously the ((3 5 3 5)) should, first of all, be mentioned. Just as ((3 4 3 4)), this solid is bounded by rhombs, and thus has equal edges. The number of faces of the "rhomb-triakontahedron" is 30 and it has 32 vertices; the 20 trihedral ones coincide with those of an R12 or {5,3}, while the other 12 form an R20 or {3,5}. The angles of the rhomb are 63.43° (arccos $1/\sqrt{5}$) and 116.57° (arccos $-1/\sqrt{5}$). The ratio of the diagonals is 2 : ($\sqrt{5}$ + 1). The long diagonals form the edges of a {3,5}, the short ones those of a {5,3}. This solid is, therefore, analogous to the ((3 4 3 4)), except for the fact that it is not space-filling; the dihedral angles are 144°, which is not a divisor of 360°. Nevertheless it is an intriguing question whether going round twice could possibly result in a kind of "second-order" space-filling; going further into this question would, however, lead us too far.
The ((5 6 6)) and the ((3 10 10)) can be imagined to originate from R12 and R20 in analogous ways as the ((4 6 6)) and the ((3 8 8)) from R8 and R6, as discussed in the previous section, namely by locating obtuse pyramids on the faces. The ((5 6 6)), the "pentakisdodecahedron", is bounded by 60 isosceles triangles, the sides of which are in the ratio of 6 : 6 : (9 – $\sqrt{5}$). The base angles are given by: $\cos \alpha = (9 - \sqrt{5})/12$, $\cos \beta$

$= (9\sqrt{5} - 7)/36$; $\alpha = 55.69°$, $\beta = 68.62°$. The top angles join together in 12 pentahedral vertices (those of an R20), the base angles in 20 hexahedral vertices (those of an R12). There are 90 edges.

In analogy to the ((5 6 6)) described above, the ((3 10 10)), the “triakisicosahedron”, is also characterized by 60 isosceles triangles, 12 + 20 vertices, and 90 edges. In this case, however, it are the faces of the R20 at which pyramids are placed (three-sided ones). The top-angles of the triangles are 119.04° (arccos $-0.06(1+\sqrt{5})$) and the base-angles 30.48° (arccos$(15+\sqrt{5})/20$); the edges are in the ratio $10 : 10 : (15+\sqrt{5})$.

In Figure 4.13 the three U2's mentioned above are depicted.

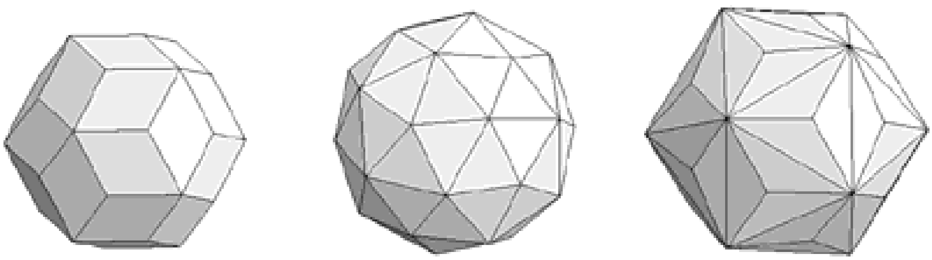


Figure 4.13 The rhomb-triakontahedron ((3 5 3 5)), the pentakis- dodecahedron ((5 6 6)) and the triakisicosahedron ((3 10 10))

The ((3 4 5 4)), the “deltoidhexacontahedron” is, as its name indicates, bounded by 60 deltoid-shaped faces (see Figure 4.14). It contains 20 trihedral, 30 tetrahedral and 12 pentahedral vertices, in total 62 (coinciding, respectively, with those of an R12, a (3 5 3 5) and an R20. The polyhedron has 120 edges. The sides of the tetragons are in a ratio of: $1 : 1 : (7+\sqrt{5})/6 : (7+\sqrt{5})/6$ (= 1,54).

The ((4 6 10)) is called the “hexakisicosahedron” after its number of faces, 120. Its faces are scalene triangles, whose sides are in a ratio of $(14+2\sqrt{5}) : (9+3\sqrt{5}) : 10$ (= 18.47 : 15.71 : 10). It has, again, 62 vertices, related to simpler solids in a similar way as with the ((3 4 5 4)), but, of course, with different relative magnitudes. There are now 20 hexahedral, 12 decahedral and 30 tetrahedral angles. Figure 4.15 shows this polyhedron.

Finally we consider the “pentagonhexakontahedron”, the ((3 3 3 3 5)), composed of 60 pentagons. These pentagons have a similar symmetry as the ((3 3 3 3 4)), but the top angle of the pentagon is now situated in a pentahedral angle. The four equal angles are 118.14°, the top-angle 67.45°. The two long sides are in a ratio of 1.750 : 1 to the three short ones. This solid also exists in two modifications, forming each other's mirror image; one of these two is shown in Figure 4.16.

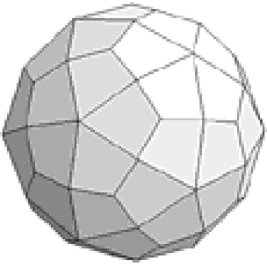


Figure 4.14
deltoidhexakontahedron
((3 4 5 4))

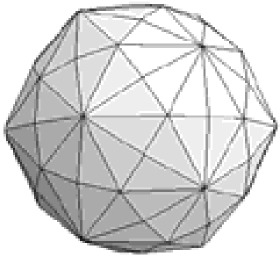


Figure 4.15
hexakisicosahedron
((4 6 10))

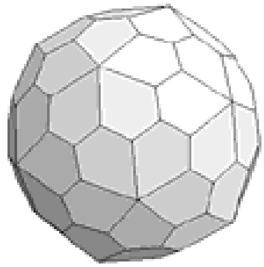


Figure 4.16
pentagonhexakontahedron
((3 3 3 3 5))

4.10 SERIES

In view of the analogy between uniform polyhedra of the first and the second kind, the U2's can be arranged into similar series as the U1's, which were discussed in Chapter 3 (Figure 3.22). The relation depicted in Figure 3.21 also holds, but now for dually transformed quantities. This means that, instead of considering V/m we should now take F/n, the number of faces divided by the number of corners per face, and, instead of r/*l* we should take a measure for the dihedral angle. The quantity r/*l* was related to the angle φ at which we see, from the midpoint, an edge, according to: $\cos \varphi = 1 - l^2/2r^2$, while the dihedral angle of the U2 equals 180° – φ. In both cases φ gives an impression of the degree in which the polyhedron approaches to a sphere.

A more generally applicable measure for the approach to the sphere is the index of compactness, ci, mentioned in § 3.18, which is the ratio between the area of the polyhedron to that of a sphere with the same volume. This index can be calculated for arbitrary solids. Its values for the U2's are, in general, comparable to those for the U1's (see Table). The ((4 6 10)) excels with ci = 0.986; this solid has only a 1.4 % greater area than a sphere with the same volume! This is, of course, related to the fact that it has the greatest number of faces, viz. 120, though the correlation between the number of faces and the index of compactness is perfect for the Platonic solids only. The ((4 6 10)) is followed by ((3 4 5 4)) and ((3 3 3 3 5)), both with ci = 0.982.

ci-values:

((3 4 3 4))	.9047	((5 6 6))	.9795
((3 6 6))	.8644	((3 10 10))	.9680
((4 6 6))	.9447	((3 4 5 4))	.9819
((3 8 8))	.9244	((4 6 10))	.9857
((3 4 4 4))	.9546	((3 3 3 3 4))	.9556
((4 6 8))	.9691	((3 3 3 3 5))	.9816
((3 5 3 5))	.9609		

The "parameter of unroundness" λ, also defined in § 3.18, has for a U2 the same value as for the U1 related to it.
The series shown in Figure 3.21 all ended (except the prisms and the antprisms) in a plane tessellation. This is also the case with the U2's; the three regular tessellations (3 3 3 3 3 3), (4 4 4 4) and (6 6 6) remain, after transformation, identical in ((6 6 6)), ((4 4 4)) and ((3 3 3 3 3 3)); the eight Archimedean tessellations obtain, of course, a different shape. These are represented in Figure 4.17.

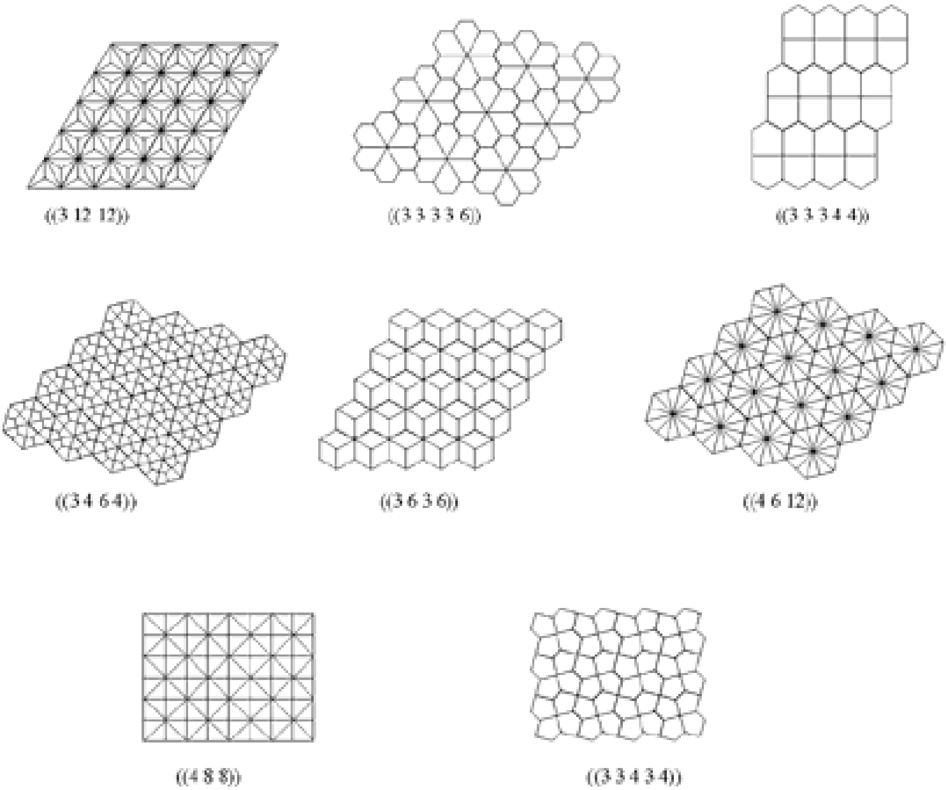


Figure 4.17 Uniform plane tessellations of the second kind

5

REGULARITY WITH STARS (POINSOT-SOLIDS)

5.1 INTRODUCTION

The description of regular and semi-regular polyhedra could have been finished with the previous Chapter if we would stick to the conventional definition of a polygon in which the sides of the polygon are not allowed to intersect between its corners.
It is, however, possible to drop this requirement; then the number of regular polygons, and also the number of regular and semi-regular polyhedra, increases considerably.
The pentagram, shown in Figure 5.1, can be considered as a regular polygon with corners A, B, C, D and E, since its sides AB, BC, CD, DE and EA are equal, as well as the angles at its corners. The points of intersection of these sides cannot be considered as corners; their function is analogous to the points of intersection of the (elongated) sides of a "normal" polygon.

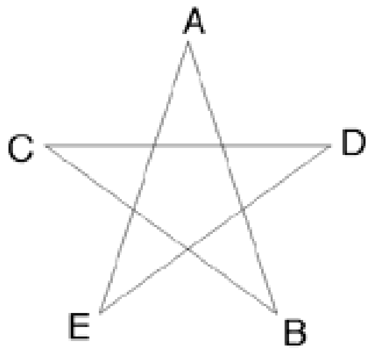


Figure 5.1 Pentagram $\{5_2\}$

This polygon and similar ones are called "higher-order" or "higher-density" polygons. When, in a regular positioning of the corners, we connect a corner with the next one to obtain the edges, we do not take the adjacent corner but, in the example mentioned, the second one. In this way the star-shaped pentagon, the pentagram, is born as a second-order pentagon, indicated by $\{5_2\}$.

How big are the angles in such a higher-order polygon? Starting from A, the edges run to B and to E, so at both sides a corner is being skipped. The angle at A, therefore, comprises two less segments of the circum-circle, and thus it is not:

$$\frac{1}{2}\cdot\frac{5-2}{5}\cdot 360° = 108° \quad \text{but: } \frac{1}{2}\cdot\frac{5-2\times 2}{5}\cdot 360° = 36°.$$

In general, the angle of an n-gon of the order a, $\{n_a\}$ equals (n – 2a)/n·180°, and the sum of the n angles is (n – 2a)·180°.
Other examples of higher-order polygons are (see Figure 5.2) the $\{8_3\}$ and the $\{10_3\}$.

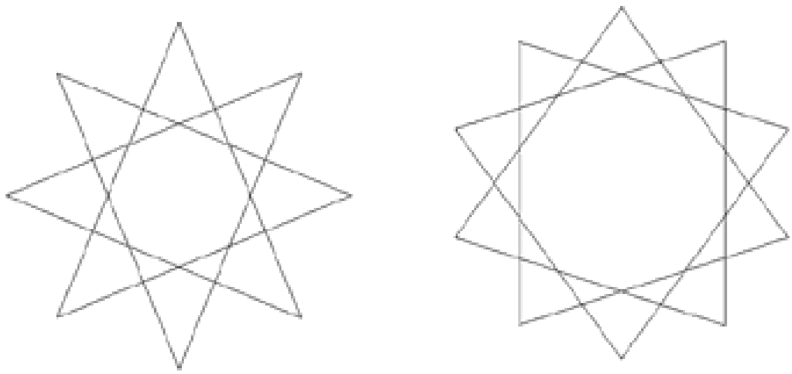


Figure 5.2 Higher-order polygons: $\{8_3\}$ en $\{10_3\}$

In general the order (or the density) of a polygon can be defined as the number of sides which are intersected by a half-line from its midpoint; this line should not pass through one of the corners. Expressed in a different way: When the polygon is "blown-up" from its centre, until all its sides become parts of a circle, the circle is covered as many times as the order of the polygon indicates.
Obviously the existence of higher-order polygons results in an extension of the definition of a polyhedral angle. When we construct a plane through each of the sides of the star-pentagon in such a way that these planes have a common point of intersection, then these planes form together a second-order pentahedral angle or a 5_2-hedral angle (with functionality 5/2). When the line from this point of intersection perpendicular to the plane of the pentagram, passes through its centre, we have a regular 5_2-hedral angle.
For regular polyhedral angles of a higher order, the formula in Chapter 1 (form. 1.5) is again valid, which gives the relation between dihedral angles and edges:

$$\cos\varphi = \frac{\cos\alpha - p}{\cos\alpha + 1}$$

in which $p = 1 + 2 \cos (360°/m)$, but now we replace m by m_b, in which b is the functionality of the polyhedral angle. We thus find for 5_2, 8_3, and 10_3 polyhedral angles, values for p of, respectively $-(\sqrt{5} - 1)/2$, $(1 - \sqrt{2})$, and $(3 - \sqrt{5})/2$.
Further on in this chapter we shall consider the consequences of both of these extensions, of the polygon as well as of the polyhedral angle, with respect to the regular polyhedra. Then also the quantity "order" of polyhedra will be discussed. This will be done in an analogous way as for polygons; when we pass, starting from the mid-point of a polyhedron and travelling outward, c times a face (not in vertices or on edges), the order of a polyhedron is called c. In doing so, the central part of a face with the order a, is counted a times, a part of a face which is separated by one edge from the center part, (a – 1) times, etc.

5.2 ANALYSIS

Let the number of faces of a higher-order polyhedron be F, the number of vertices V and the number of edges E. The faces are n-gons of the order a, the vertices form m-hedral angles of the order b.
Independent of the values of a and b, the well-known relations between F, V, E, n and m are valid:

$$F \cdot n = V \cdot m = 2 \cdot E \, .$$

Euler's formula is also valid for higher-order polyhedra, albeit in a modified shape. In the original formula,

$$V + F = E + 2 \, ,$$

V has now to be replaced by b·V and F by a·F, in other words: vertices as well as faces are counted as many times as their order indicates. The constant 2 is, moreover, replaced by 2·c (c is the order or the density of the polyhedron). Euler's rule now becomes:

$$b \cdot V + a \cdot F = E + 2 \cdot c \, .$$

The proof of this extended formula of Euler will not be given here.
Combination of this formula with the relations given above, results in:

$$F \cdot n = V \cdot m = 2 \cdot E = \frac{2 \cdot c}{a/n + b/m - 1/2}$$

For a = b = c = 1 these relations are reduced to those already known from Chapter 1.
When we take these relations as a starting point, we can try out, which combinations result in a possible polyhedron. Let us first consider the polygon $\{5_2\}$. How many possibilities exist to join a number of these polygons into a first-order vertex? As far

as the value of the top-angle (36°) concerns, m-values of 3 up to 9 could be considered. To investigate these possibilities, we can write the equations as follows:

$$2\cdot(c/V) = b - (m/n) \cdot (n/2 - a)\ ; \quad F = V\cdot(m/n)\ .$$

For the case under investigation with n = 5, a = 2, b = 1 this becomes:

$$2\cdot(c/V) = 1 - m/10\ , \qquad F = (V/5)\cdot m\ .$$

For various values of m we find several potential possibilities to form a polyhedron. The question is now, whether all these possibilities lead to really existing polyhedra. When we try them out, most of them disappear. This is not surprising, since we have only considered the conditions for joining faces at vertices, such as for the acute angle A of Figure 5.1, but the adjacent corners C and D should also be able to fit into a vertex with other faces.

From further analysis an extra criterion can be derived, namely that the vertices of regular higher-order polyhedra should coincide with those of a first-order, Platonic, polyhedron. Thus V should, for pentagons, be 12 or 20. The same should, of course, hold after dual exchange, so that F can only be 12 or 20. According to this criterion the only remaining possibilities are: m = 3 with c = 7, and m = 5 with c = 3. These combinations result in the polyhedra $\{5_2,3\}$ and $\{5_2,5\}$ or, in another notation, $(5_2\ 5_2\ 5_2)$ and $(5_2\ 5_2\ 5_2\ 5_2\ 5_2)$.

Systematic analysis of the other possibilities only results in the dually related pair: $\{3\ ,\ 5_2\}$ and $\{5\ ,\ 5_2\}$ or, respectively, $(3\ 3\ 3\ 3\ 3)_2$ and $(5\ 5\ 5\ 5\ 5)_2$, which can, of course, also be denoted as $((5_2\ 5_2\ 5_2))$ and $((5_2\ 5_2\ 5_2\ 5_2\ 5_2))$. A survey of the parameters of these four solids, together with the related first-order Platonic solids, is given in the table below.

	n	a	m	b	Z	H	R	c	name
$\{5_2, 3\}$	5	2	3	1	12	20	30	7	great stellated dodecahedron
$\{5_2, 5\}$	5	2	5	1	12	12	30	3	small stellated dodecahedron
$\{3, 5_2\}$	3	1	5	2	20	12	30	7	great icosahedron
$\{5, 5_2\}$	5	1	5	2	12	12	30	3	great dodecahedron
$\{3, 5\}$	3	1	5	1	20	12	30	1	icosahedron
$\{5, 3\}$	5	1	3	1	12	20	30	1	dodecahedron

The four higher-order regular polyhedra are called the “Kepler-Poinsot solids”. Kepler described the first two of them; the other two were first discovered by Poinsot (1810), who presented the complete collection of higher-order regular polyhedra as an addition to the five Platonic solids.

5.3 THE GREAT STELLATED DODECAHEDRON {5_2,3}

Though this polyhedron is, of the four Kepler-Poinsot solids, the most closely related to the dodecahedron {5 ,3}, it can easiest be thought to originate from an icosahedron {3,5} by stellation. When the edges of the R20 are extended into both directions, until they intersect three by three, at each face of the icosahedron three-sided pyramids are built which together form a 20-pointed star (see Figure 5.3). Each of the faces of such a pyramid is part of a pentagram {5_2}; the middle part of this pentagram is formed by the intersection of the icosahedron with a plane through five vertices adjacent to a vertex. These 12 pentagons enclose a regular dodecahedron, which is the core of the great stellated dodecahedron.

From the analysis it follows that the order of the {5_2,3} is 7: this can also be seen by travelling along a half-line from its centre; we then pass four faces, namely three times a double-counting central part of a pentagram and once an outer part.

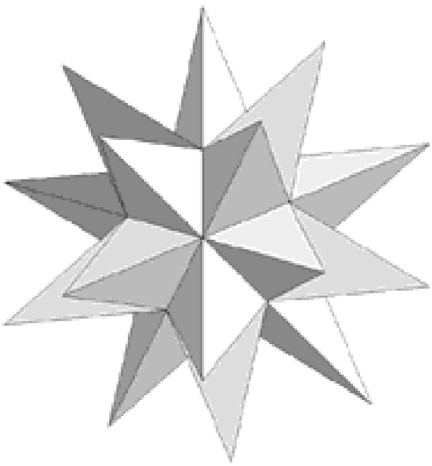


Figure 5.3 Great stellated dodecahedron

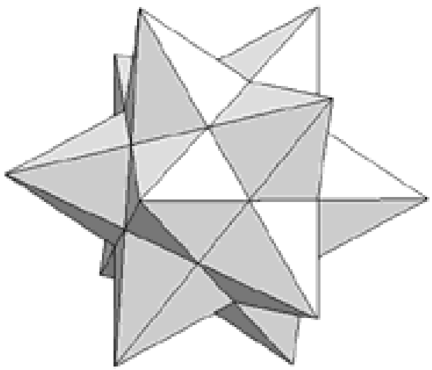


Figure 5.4 Small stellated dodecahedron

5.4 THE SMALL STELLATED DODECAHEDRON {5_2,5}

In the same way as the {5_2,3}, as discussed in the previous section, could be thought to originate from the {3,5} by stellation, the small stellated dodecahedron {5_2,5} can be formed from the dodecahedron {5,3}. In this case, however, the planes of their faces coincide: the {5_2,5} is created by extending the 12 faces of the {5,3} (see Figure 5.4). The pentagons are thereby extended to pentagrams {5_2}, five of which meet in a vertex. The number of vertices equals the number of faces, namely 12, which is a consequence of the fact that m = n, which we, so far, only encountered with the tetrahedron (m = n = 3, V = F = 4).

The order of $\{5_2,5\}$ is 3; starting from its centre we meet two faces, once we pass through the double-counting central part of a pentagram and once through one of the points.
The core of the small stellated dodecahedron is formed by the dodecahedron mentioned before. The vertices coincide with those of an icosahedron.
When we compare the $\{5_2,3\}$ with the $\{5_2,5\}$ on the basis of their common core, the dodecahedron, with both of which they have the planes of their 12 faces in common, the names "great" and "small" are clear: the great one has a $(5\sqrt{5} + 11)/2 = 11.1$ times longer edge then the inscribed dodecahedron, and the small one $(\sqrt{5} + 2) = 4.24$ times greater. The edges of the great and the small ones are, therefore, in the ratio 2.62 : 1.

5.5 THE GREAT ICOSAHEDRON $\{3 , 5_2\}$

This solid is narrowly related to the icosahedron $\{3,5\}$ (the same number of vertices, faces and edges), and it can be derived from $\{3,5\}$ in two different ways, namely either via its vertices or via its faces.
Using the vertices, we can inscribe a $\{3,5_2\}$ into a $\{3,5\}$: its edges are the spatial diagonals of $\{3,5\}$ which do not pass through the centre, as indicated in Figure 5.5 for two series of three edges, each forming a face of $\{3,5_2\}$. In each vertex five of the 20 faces meet together in a second-order polyhedral angle

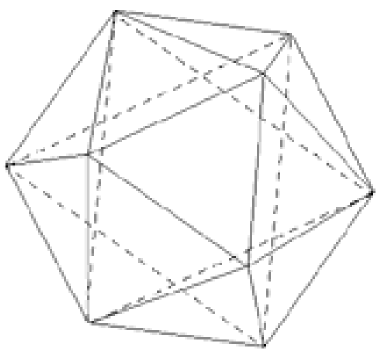


Figure 5.5 Two faces of $\{3 , 5_2\}$ in $\{3 ,5\}$

If we start off from the faces of an icosahedron, then we obtain a circumscribed $\{3,5_2\}$, which has the $\{3,5\}$ as a core. If we take a face of $\{3,5\}$ and intersect it with all other faces (except the opposite one) then we obtain a pattern as indicated in Figure 5.7. This figure contains 18 line segments, which form the intersections with the 18 faces. In this figure we can recognize the compounds of five octahedra and of five tetrahedra. The inner triangle is the face of the original icosahedron, the outer triangle is the face of the great icosahedron. From the figure it is clear that the order of $\{3,5_2\}$ is 7. The only parts of the faces which are visible from the outside, are the nine triangles which are adjacent to the outer circumference.

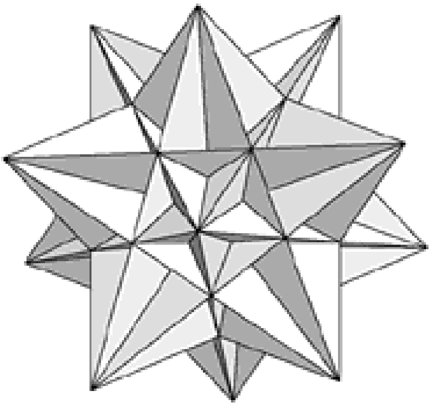


Figure 5.6 Great icosahedron $\{3,5_2\}$

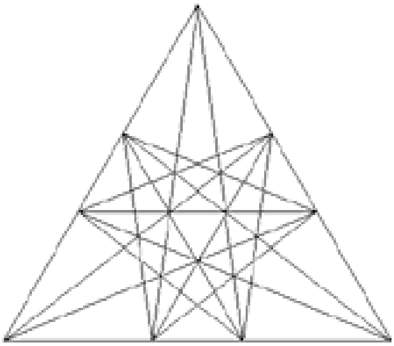


Figure 5.7 Face of $\{3,5_2\}$ with intersections

When we compare $\{3,5_2\}$ with $\{5_2,5\}$, then it appears that not only their vertices, but also their edges can coincide. The great icosahedron can, therefore, be very narrowly enclosed by the small stellated dodecahedron!

5.6 THE GREAT DODECAHEDRON $\{5,5_2\}$

A further look at the icosahedron $\{3,5\}$ reveals that it contains 12 $\{5\}$-shaped diagonal faces. These faces form together the great dodecahedron, which, therefore, has the same vertices and edges as the icosahedron.
Another way to create this solid, is by extending the pentagram-shaped faces of the small stellated dodecahedron $\{5_2,5\}$ to pentagons $\{5\}$ with the same corners. In this way the relation with the other three dodecahedra becomes clear; this is demonstrated in Figure 5.8.

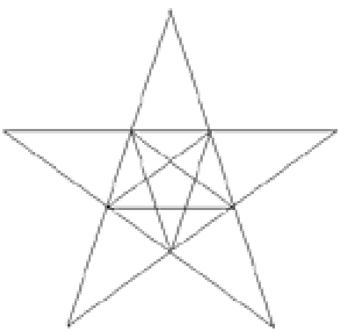


Figure 5.8 Faces of the four dodecahedra

At the inside the pentagon is situated which forms the face of the dodecahedron $\{5,3\}$. When we elongate the edges to form a pentagram $\{5_2\}$ we obtain the small stellated dodecahedron $\{5_2,5\}$. Extension of the pentagram to a pentagon $\{5\}$ creates the great dodecahedron $\{5,5_2\}$; extending the edges of this pentagon into a pentagram $\{5_2\}$ leads to the great stellated dodecahedron $\{5_2,3\}$. We cannot continue: connecting the corners of the largest pentagram to a pentagon results in incoherent faces.

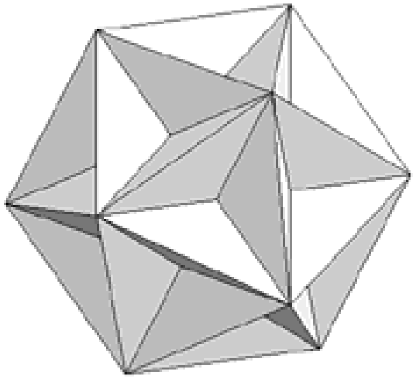


Figure 5.9 Great dodecahedron $\{5\ ,\ 5_2\}$

The outer surface of the great dodecahedron is composed of 60 isosceles triangles with angles of 108°, 36° and 36°, which, three by three, form hollow pyramids in the faces of the enveloping icosahedron; the obtuse tops of these pyramids are the vertices of the dodecahedron which forms the core of the great dodecahedron.

5.7 NINE PLATONIC SOLIDS

In this chapter we have seen that the extension of the notions “regular polygon” and “regular polyhedral angle” leads to the completion of the number of Platonic solids from 5 to 9. We also saw, though not fully proven, that further extension of this series is not possible.

In view of the narrow relationship of the four new solids with the dodecahedron and the icosahedron, it is not surprising that all ways of fitting the latter into tetrahedron, cube and octahedron, also hold for the Poinsot solids.

Our next step will now be, to investigate whether the extended definitions of regular polygons and polyhedral angles will also lead to new semi-regular solids. This will be dealt with in the following chapter.

6

SEMI-REGULARITY WITH STARS (HIGHER-ORDER ARCHIMEDEAN SOLIDS, UH's)

6.1 INTRODUCTION

At the end of the previous chapter we have asked ourselves whether the introduction of star-shaped regular polygons and polyhedral angles also results into other, higher order, Archimedean or uniform solids, as it appeared the case with the Platonic solids. This is indeed the case. The five originally known Platonic solids were supplemented by four other ones. The series of original 13 Archimedean solids (not counting the prisms and the antiprisms) grows by at least 53 new types! The same holds, of course, for the uniform polyhedra of the second kind, the dually related specimens.
These new polyhedra can be thought to originate from the "old" ones in several ways:

- by replacing their faces (five- ore higher-gons) by inscribed or circumscribed stars,
- by transforming four- or more- hedral vertices into stellated vertices,
- by truncating higher-order polyhedra such as the Poinsot solids, but also those obtained by methods mentioned above, at their vertices.
- etc.

Enough possibilities, since we meet, next to five-stars, also 8- and 10-stars. The resulting complete series is too large to be given in this book; therefore only a few examples will be given. But before giving these, a first approach will be made to investigate which higher-order Archimedean solids exist.

6.2 ANALYSIS

Analysis of the possible uniform higher-order polyhedra (UH's) goes, in principle, along the same lines as with the regular higher-order solids (§ 5.2). We have, however, now to take into account that in each of the identical vertices polygons of different kinds meet, say m_1 regular polygons $\{n_1/a_1\}$, m_2 of the type $\{n_2/a_2\}$ etc. These polygons form together an m-hedral angle of the order b ($m = \sum m_i$). The total

numbers of faces are represented by $F_!$, $F_{2,}$, F_3 etc., with $F = \sum F_{i.}$ The number of edges is E, the number of vertices V.
Since at each vertex m_i faces of the form $\{n_i/a_i\}$ meet, the resulting sum over all vertices is $V \cdot m_{i\,.}$ Now each face has been counted n_i times, so $F_i = V \cdot (m_i/n_i)$ or $F_i \cdot n_i = V \cdot m_i$. Summation over all kinds of faces results in $\sum F_i \cdot n_{i\,=}$ V/m, which equals $2 \cdot$ E. So:

$$\sum F_i \cdot n_i = V \cdot \sum m_i = 2 \cdot E$$

Combined with the extended formula of Euler :

$$b \cdot V + \sum a_i \cdot F_i = E + 2 \cdot c$$

we find the relation:

$$\frac{2c}{V} = \sum \frac{m_i}{n_i/a_i} - \frac{\sum m_i}{2} + b$$

In principle, many combinations of values for n_i, a_i, m_i and b could be substituted into this equation in order to obtain possible values for V and c, with the appropriate values for the F_i's. Even when the same restrictions are applied as in Chapter 3 for the simpler Archimedean solids, the number of possible combinations is very high. When such a combination leads to a non-integer value for V/c, a proper choice of c could turn V into an integer.
Many of these cases, however, do not yield a possible polyhedron. In the same way as for the regular higher-order solids, it can be proven that higher-order uniform polyhedra should possess a certain relation to first-order ones, namely that such a first-order specimen should form its core, while the vertices should coincide directly or after transformation.
As a result of this, we meet for the values of V only those which we saw already in par. 3.2; moreover, just as with the first-order uniform polyhedra, the faces are 3-, 4-, 5-, 6-, 8- or 10-gons.
Further on in this chapter we shall consider a number of examples, chosen from the many possibilities, while using the classification as given in 6.1. In particular, Archimedean higher-order polyhedra of the first kind will be dealt with, though, incidentally, also some of the second kind (duals) will be mentioned.

6.3 STELLATION OF FACES

First of all an example from the world of the prisms. The eight-sided prism (4 4 8) can be transformed into a higher-order prism by inscribing 3rd order eight-pointed stars into the 8-gons; the height of the prism will, as a matter of fact, change. In this way we find the (4 4 8_3), shown in Figure 6.1. What is the order of this prism?

According to the analysis in 6.2 we find with V = 16, $n_1 = 8$, $n_2 = 4$, $a_1 = 3$, $a_2 = 1$, $m_1 = 1$, $m_2 = 2$, $F_1 = 2$, $F_2 = 8$, that E = 24 and c = 3; its order is, therefore, 3. This is quite understandable: from a point in the middle a line may pass the 8-pointed star through its centre, or it may intersect three squares.

In a somewhat more elaborate, but possibly more transparent way, we arrive at the same result: The corners of the third-order octagon form angles of 45° (see § 5.1):

$$\frac{1}{2} \cdot \frac{8 - 3 \cdot 2}{8} \cdot 360° = 45°$$

The sum of the angles in a vertex of ($4\ 4\ 8_3$) is, therefore, 90° + 90° + 45° = 225°, and the angular deficit is 360° - 225° = 135°. The sum of the angular deficits at all vertices mounts to V·135° and this should be an integer multiple of 720° , or c·720°, in which c is the order of the polyhedron or the number of times that the circum-sphere is covered by the polyhedron. In this case it appears that the smallest possible value of c is 3 with V = 16. From Euler and the other relations the values for the F's and R can de derived.

A similar example is the origination from (3 3 3 9) of ($3\ 3\ 3\ 9_4$), demonstrated in Figure 6.2.

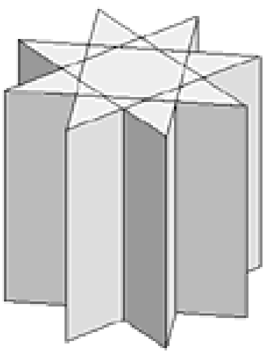


Figure 6.1 Prism ($4\ 4\ 8_3$)

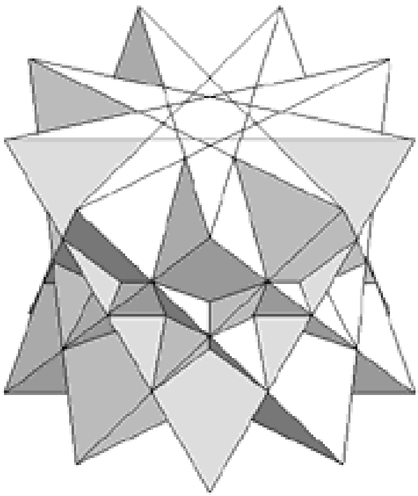


Figure 6.2 Antiprism ($3\ 3\ 3\ 9_4$)

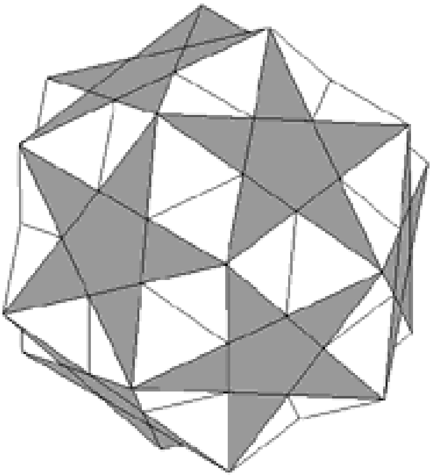


Figure 6.3 $(3\ 5_2\ 3\ 5_2\ 3\ 5_2)$

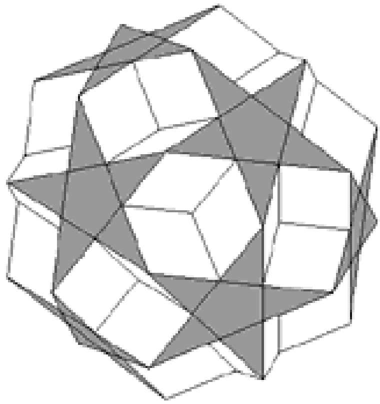


Figure 6.4 $(5\ 5_2\ 5\ 5_2)$

Another simple example is the substitution of the pentagons in the regular dodecahedron by pentagrams $\{5_2\}$. Between the three acute angles "holes" are now formed, which are filled by equilateral triangles. The result is $(3\ 5_2\ 3\ 5_2\ 3\ 5_2)$, with 20 vertices, 12 pentagram faces and 60 edges (Figure 6.3). According to the formula, the order of this polyhedron is c = 2.

The next example in this series is $(5\ 5_2\ 5\ 5_2)$, which originates by replacing the pentagons in (3 5 3 5) by $\{5_2\}$ pentagrams. The triangles disappear, and 12 new pentagons are formed (Figure 6.4). From the formula in 6.2 it follows that, with V = 30, the order c = 3.

Finally we look at two UH's which can be derived from (3 8 8), namely $(3\ 8_3\ 4\ 8_3)$ and $(3\ 8_3\ 8_3)$, which are formed, respectively, by inscribing and circumscribing 3rd order octagons in and round the octagons of the (3 8 8). In both cases the number of vertices remains V = 24. With the $(3\ 8_3\ 4\ 8_3)$, next to the eight triangles and the six octagons, six squares are being formed (Figure 6.5). The $(3\ 8_3\ 8_3)$ is composed of eight triangles and six third-order octagons (Figure 6.6)

Next to each of the UH's of the first kind (UH1's) described so far, a dually related one of the second kind (UH2) exists. This can, just as in Chapter 4, be constructed by applying tangent planes to the circum-sphere at the vertices.The faces of these U2's are non-regular 3-, 4- or 6-gons. Their shape is often difficult to recognize as a result of the various intersections of the faces. Some of the UH2's, belonging to the UH1's mentioned before, are shown in Figures 6.7 to 6.10.

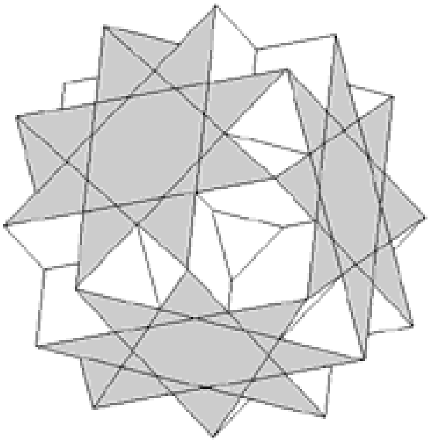


Figure 6.5 $(3\ 8_3\ 3\ 8_3)$

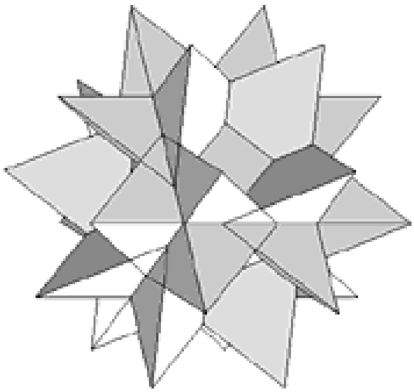


Figure 6.6 $(3\ 8_3\ 8_3)$

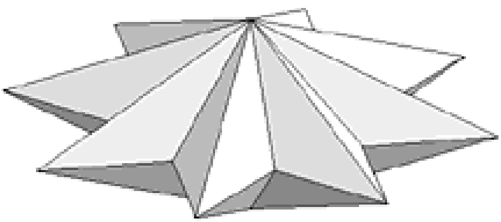


Figure 6.7 $((4\ 4\ 8_3))$

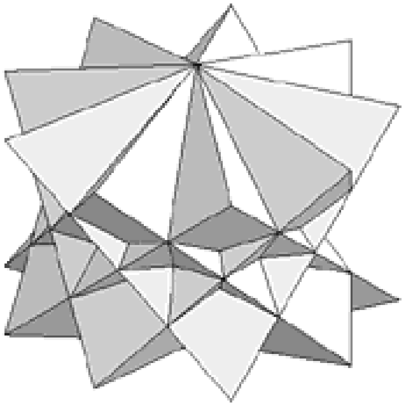


Figure 6.8 $((3\ 3\ 3\ 9_4))$

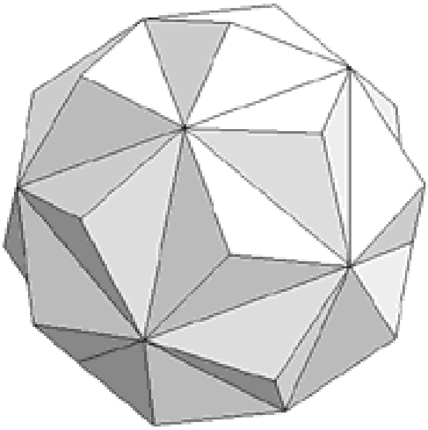


Figure 6.9 $((3\ 5_2\ 3\ 5_2\ 3\ 5_2))$

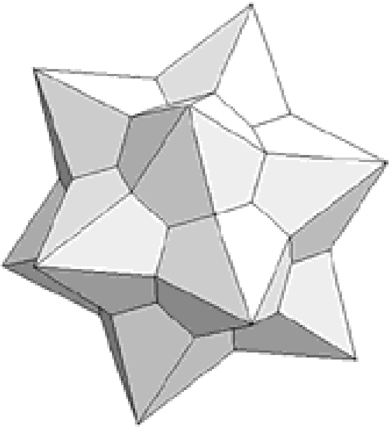


Figure 6.10 $((5\ 5_2\ 5\ 5_2))$

6.4 STELLATION OF VERTICES

Just as faces of a UH1 can be transformed into stars, its vertices can be substituted by higher-order polyhedral angles. This is easiest for regular polyhedral angles, for instance the transformation of a first-order into a second order pentahedral angle, which changes the regular icosahedron into a great icosahedron.

With the uniform or half-regular polyhedra the vertex figures are, however, never regular; stellation of the vertices will, therefore, lead to different situations. Even if we consider a regular vertex figure such as a square (in a tetrahedral angle), its regularity is lost upon stellation into a tetragon consisting of two parallel edges and two diagonals.

As long as we consider uniform polyhedra of the first kind and their transformations into higher-order ones, we do not meet the transformation of squares into stars, but we have to consider how we can change a regular tetrahedral angle into a star-shaped one! The very first example of this is a Platonic solid, the octahedron. If we replace its vertex by a higher-order tetrahedral angle, as schematically indicated in Figure 6.11, then the (3 3 3 3) is transformed into a $(3\ 4\ 3\ 4)_2$. The position of vertices, edges, and four of the eight faces remains unchanged, while four triangles disappear and three new squares are formed. The index 2 in the notation signifies, for the time being, that the polyhedral angles are of the 2nd order as a result of the stellation, in analogy to the nomenclature introduced in § 5.2.

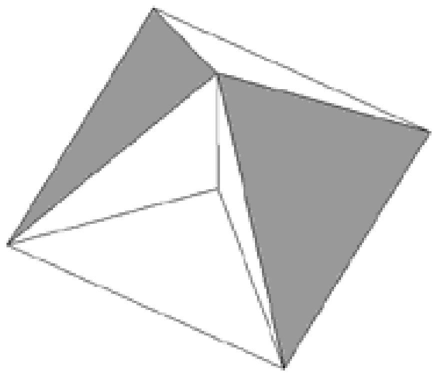
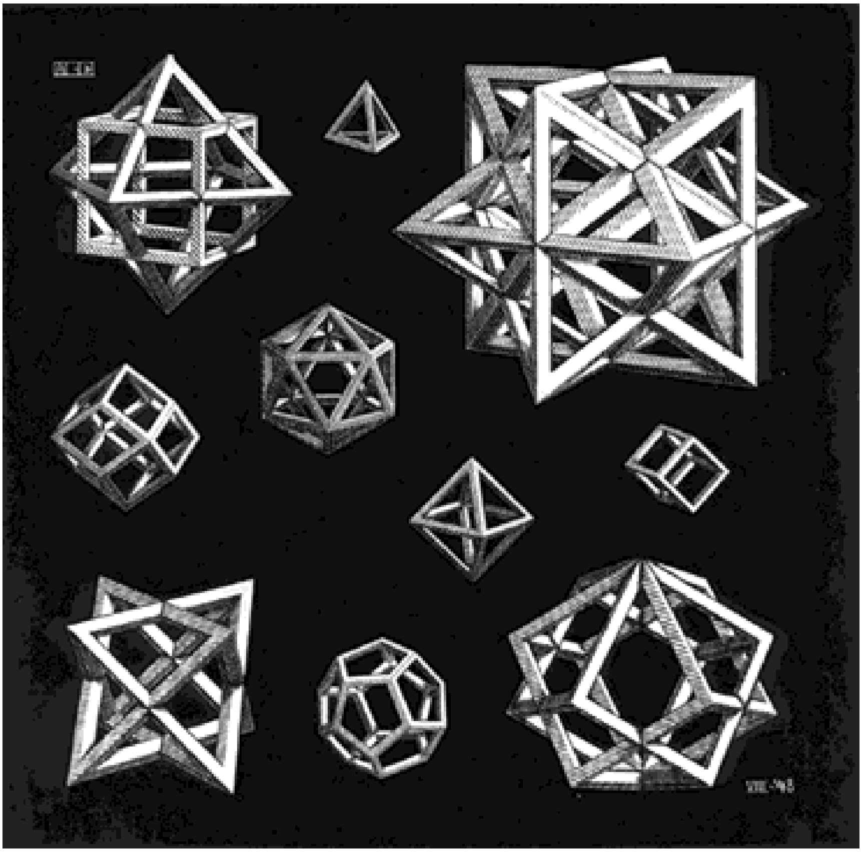


Figure 6.11 $(3\ 4\ 3\ 4)_2$

If we now apply the formulas of § 6 2 on this polyhedron, then we find, with $n_1 = 3$, $m_1 = 1$, $n_2 = 4$, $m_2 = 2$, $b = 1$: $V/c = 12/7$. Since, from the narrow relation to the octahedron, it is clear that $V = 6$, it follows that $c = 7/2$, This value is, as a non-integer, rather unusual. Let us consider the choice of b in somewhat more detail. b is the number of times that the circumscribed circle is passed through during a tour round the vertex, so ABCDA (Figure 12a). If we turn to the right at B, to the left at C and D (shortest route), and to the right at A, then the arcs AB, BD and CA are each passed through once, but CD three times; the order will then be: $\frac{1}{4} + \frac{1}{4} + 3{\cdot}\frac{1}{4} + \frac{1}{4} = 3/2$. This is, however, no longer valid when we replace the starting square by a differently shaped tetragon, e.g. a narrow rectangle. Moreover, the change of direction during the tour seems less logical. When we perform a tour during which we always turn to the right, also at C (Figure 12b), then we pass between C and D $\frac{3}{4}$ of the circle; the order then becomes: $\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + \frac{1}{2} = 2$, which is analogous to the value for other types of stellation, and which is, moreover, valid for deviations from the square shape. So we arrive again at c = 7/2. To make this value more acceptable, we can follow the same reasoning as with the tetragon: two of the four triangular faces cover the sphere each for 1/8, the other two not for 1/8 but for the remainder of the sphere, so each for 7/8. Together with the three half spheres for the square-shaped faces we indeed arrive at c= 7/2 !



Sterren (Stars), © 1948 M.C. Escher / Cordon Art – Baarn – Holland.

A special feature of $(3\ 4\ 3\ 4)_2$ is the impossibility to define an inner- and an outer side. If we make a tour over its surface round one of its vertices (Figure 6.12), than we pass from the outside into its inside. This is, in itself, not very special, but when we realize that this is also the case with the other vertices, it becomes clear that to this polyhedron

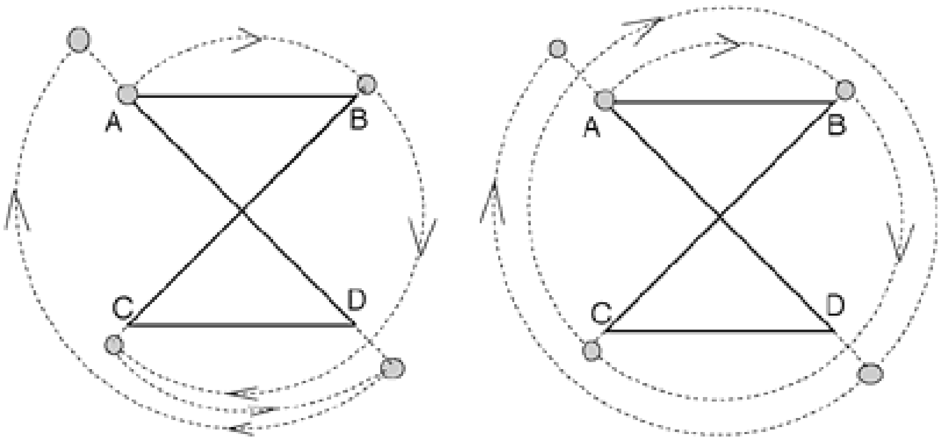


Figure 6.12 2 Tours round a vertex

no outer surface or inner surface can be assigned; in other words: it is a Möbius surface, analogous to the one formed by a strip of paper, one end of which is glued to its other end after rotating it by 180°.

Analogous to the creation of $(3\ 4\ 3\ 4)_2$ by "stellation" of the tetrahedral angles, from every uniform polyhedron U1 containing tetrahedral angles, a UH2 can be formed , or even two, because, when the tetrahedral angle is not regular, its diagonals can be combined with two opposing edges in two different ways (Figure 6.13). In the same way (3 4 3 4) can be transformed into $(3\ 6\ 3\ 6)_2$ and $(4\ 6\ 4\ 6)_2$, as indicated in Figures 6.14 and 6.15.

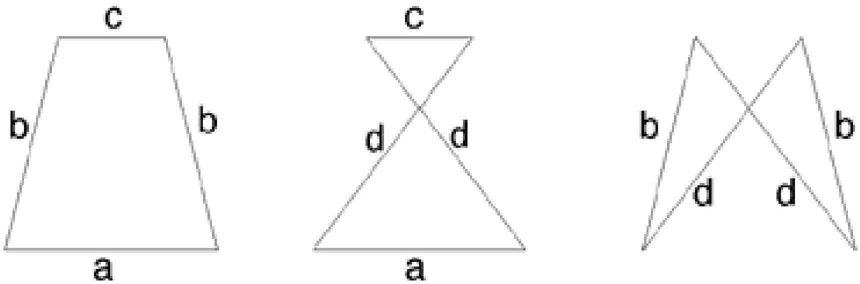


Figure 6.13 Formation of $(a\ d\ c\ d)_2$ and $(b\ d\ b\ d)_2$ from (a b c b)

From (3 5 3 5) both $(3\ 10\ 3\ 10)_2$ and $(5\ 10\ 5\ 10)_2$ can be derived (Figures 6.16 and 6.17).

The (3 4 4 4) produces the $(3\ 8\ 4\ 8)_2$ and the $(4\ 8\ 4\ 8)_2$ (Figures 6.18 and 6.19); the (3 4 5 4) the $(4\ 10\ 4\ 10)_2$ and the $(3\ 10\ 5\ 10)_2$ (Figures 6.20 and 6.21).

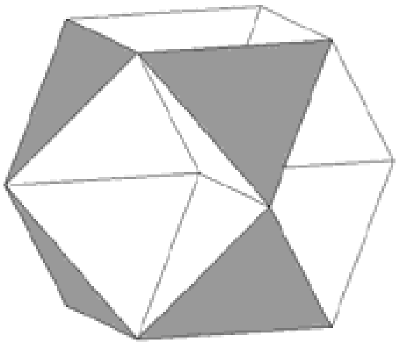


Figure 6.14 $(3\ 6\ 3\ 6)_2$

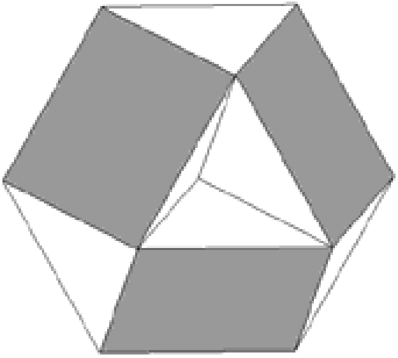


Figure 6.15 $(4\ 6\ 4\ 6)_2$

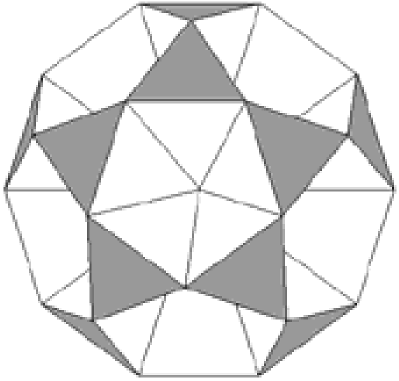


Figure 6.16 $(3\ 10\ 3\ 10)_2$

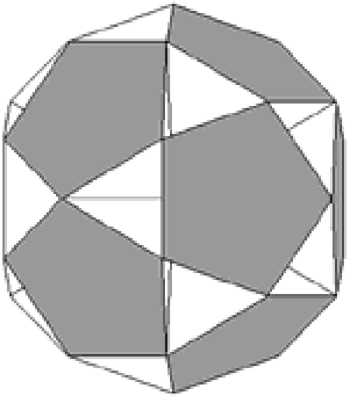


Figure 6.17 $(5\ 10\ 5\ 10)_2$

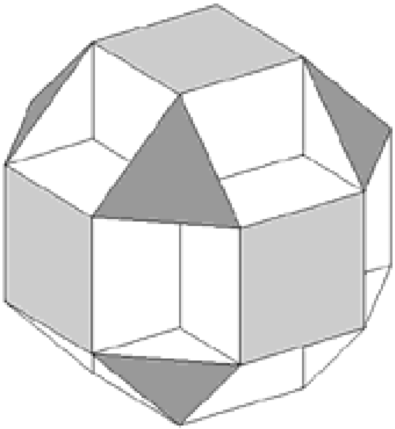


Figure 6.18 $(3\ 8\ 4\ 8)2_2$

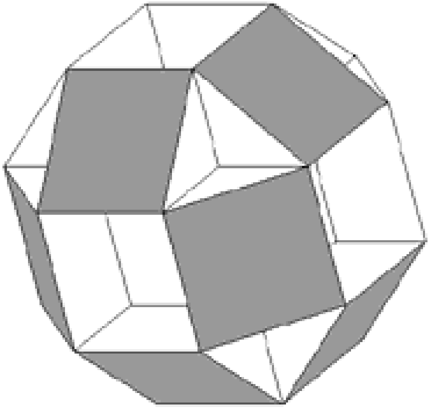


Figure 6.19 $(4\ 8\ 4\ 8)_2$

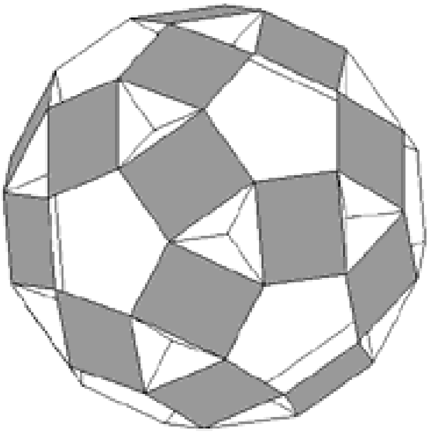


Figure 6.20 $(4\ 10\ 4\ 10)_2$

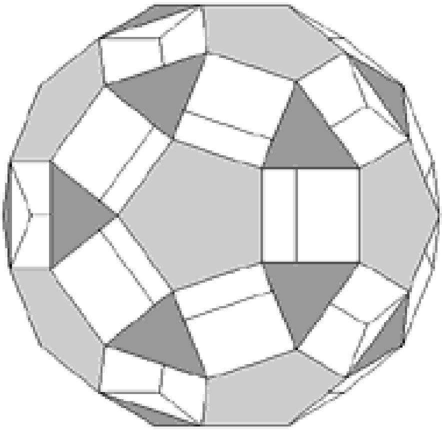


Figure 6.21 $(3\ 10\ 5\ 10)_2$

In all these cases the order b of the stellated tetrahedral angles has to be taken as 2, just as with the specimen discussed before. The Table provides further information on the UH2's mentioned in this section.

Some of them possess, as the $(3\ 4\ 3\ 4)_2$, only one side (Möbius), other ones are "normally" double-sided. This depends, a.o., on the parity of the various faces. It appears, e.g. with the $(3\ 8\ 4\ 8)_2$ that, defining the squares as "outside", we look at the "insides" of the triangles. With the related $(4\ 8\ 4\ 8)_2$ it is, however, not possible to consider all squares as "outside": two squares adjacent to a vertex are opposite to each other with respect to "outside" and "inside"; the polyhedron is, therefore, single-sided.

The schematic representations of the tetrahedral angles show that in some cases the diagonal planes pass through the origin (e.g. with $(3\ 4\ 3\ 4)_2$) ; in these cases the formation of a dual polyhedron is not possible. With other ones, such as $(4\ 8\ 4\ 8)_2$, $(4\ 10\ 4\ 10)_2$ and $(3\ 10\ 5\ 10)_2$ this restriction does not exist; their dual UH2's are shown in Figures 6.22 to 6.24.

The faces of these UH2's are, as a matter of fact, non-convex polygons; sometimes they take the shape of the polygons shown in Figure 6.13, sometimes they are arrow-shaped.

Finally, as concerns stellation in vertices, a few special cases: this procedure can also be applied to the UH1's discussed in the previous section. From the $(3\ 8_3\ 4\ 8_3)$ the $(3\ 4\ 4\ 4)_2$ can be derived (Figure 6.25). Also from $(3\ 5_2\ 3\ 5_2\ 3\ 5_2)$, stellation of the vertices results in a new UH1, $(5\ 5_2\ 5\ 5_2\ 5\ 5_2)_2$ (see Figure 6.26) as well as the $(3\ 5\ 3\ 5\ 3\ 5)_2$ (not shown).

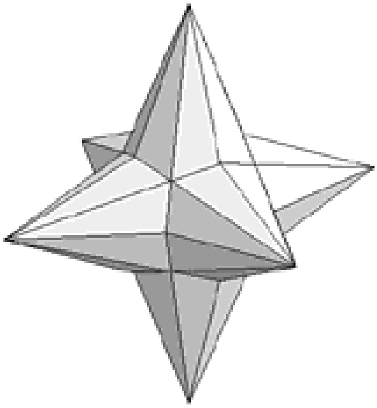


Fig.6.22 $((4\ 8\ 4\ 8))_2$

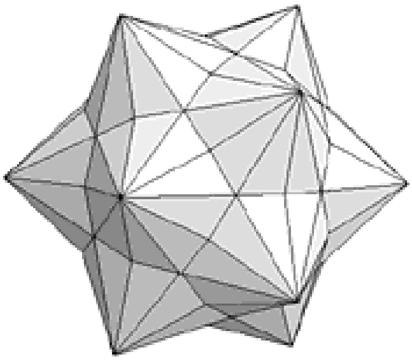


Fig.6.23 $((4\ 10\ 4\ 10))_2$

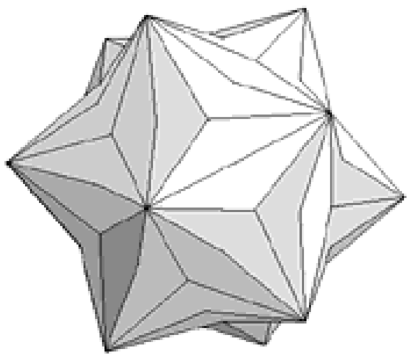


Fig.6.24 $((3\ 10\ 5\ 10))_2$

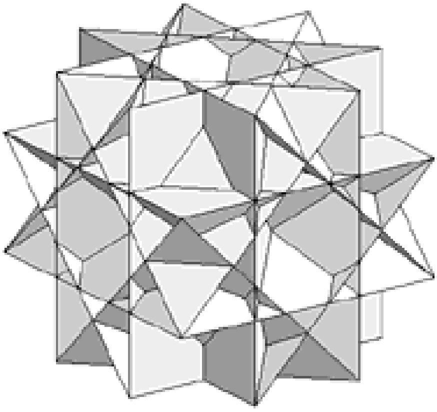


Fig.6.25 $(3\ 4\ 4\ 4)_2$

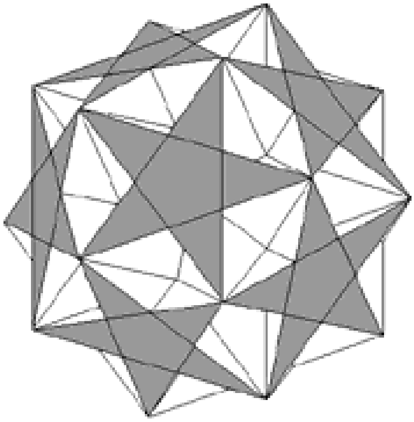


Figure 6.26 $(5\ 5_2\ 5\ 5_2\ 5\ 5_2)_2$

The final example does not really fit into this chapter, but it is interesting enough to be mentioned. It concerns a polyhedron with the notation $(6_2\ 6_2\ 6_2\ 6_2\ 6_2\ 6_2)_2$. This is not a uniform polyhedron, since its faces are non-regular polygons. The faces are, together with the polyhedron, shown in Figure 6.27. The solid looks very much like the regular dodecahedron, in which the pentagons are replaced by pyramids with equilateral triangles, pointed towards the inside. These triangles form, three by three, via elongated edges, a second-order hexagon. Twenty of these hexagons, joining six by six in the vertices, together form the special icosahedron. The most remarkable property of this polyhedron is, that it remains the same after dual transformation; it is composed of second-order hexagons, which join together in second-order hexahedral vertices. It is surprising that, next to the regular tetrahedron (at least) another polyhedron exists with this property!

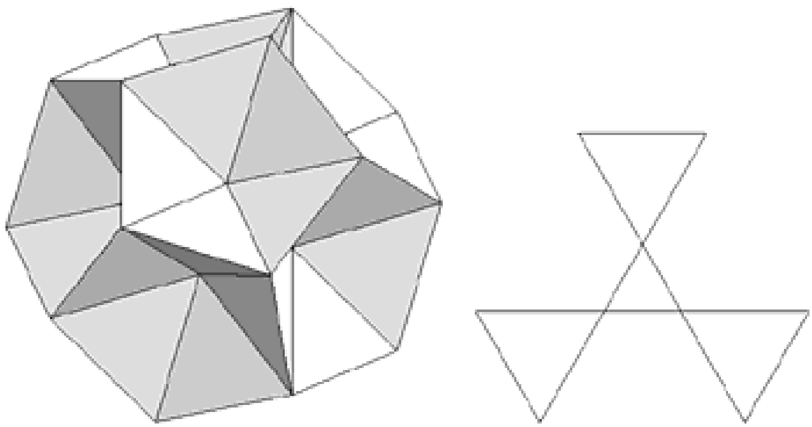


Figure 6.27 $(6_2\ 6_2\ 6_2\ 6_2\ 6_2\ 6_2)_2$ with one of its faces

6.5 TRUNCATION OF VERTICES

Of the third possibility, mentioned in 6.1, two of the simplest examples will be given, Truncation of two of the Poinsot solids results in UH's, namely with the great dodecahedron and with the great icosahedron. In both cases the vertices are fivefold and of the second order; truncation, therefore, results in $\{5_2\}$ faces.

From the great dodecahedron $\{5,5_2\}$ or $(5\ 5\ 5\ 5\ 5)_2$ the $(5_2\ 10\ 10)$ is formed, shown in Figure 6.28. It has 60 vertices, 12 $\{5_2\}$ and 12 $\{10\}$ faces and 90 edges. Its order is, just as with $\{5,5_2\}$, c = 3.

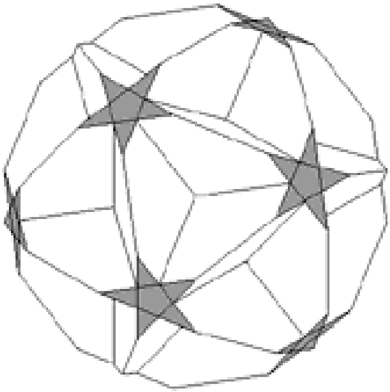


Figure 6.28 Truncated great dodecahedron (5_2 10 10)

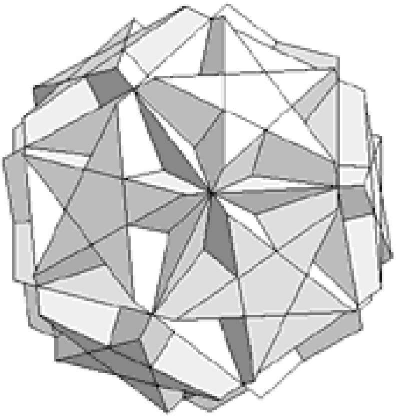


Figure 6.29 Truncated great icosahedron (5_2 6 6)

Truncation of the great icosahedron results in the (5_2 6 6) (see Figure 6.29). This polyhedron is composed of 12 pentagrams $\{5_2\}$ and 20 hexagons $\{6\}$, and it has as well 60 vertices and 90 edges. Its order is, as with $\{3,5_2\}$, c = 7.
Both UH1's have a dual UH2: the ((5_2 10 10)) and the ((5_2 6 6)), shown in Figures 6.30 and 6.31.

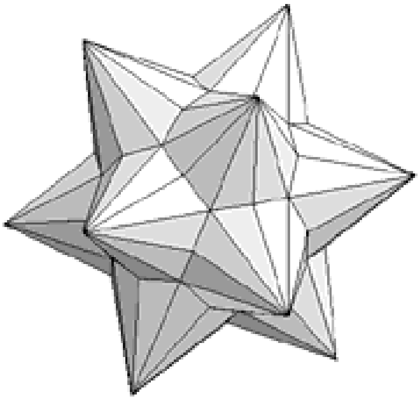


Figure 6.30 ((5_2 10 10))

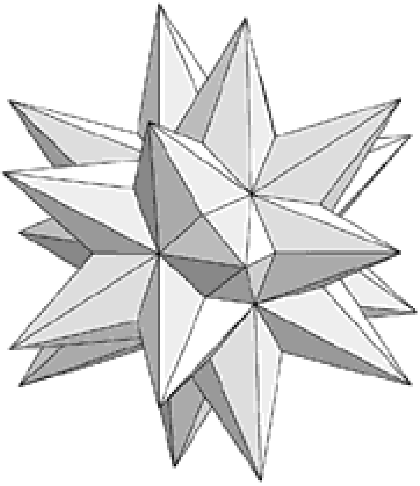


Figure 6.31 ((5_2 6 6))

In the table below the UH1's discussed so far are summarized.

notation	$(n_a)_1$ $(n_a)_2$ $(n_a)_3$	m_1 m_2 m_3	m b	V/c	V c	F_1 F_2 F_3	F	E
(4 4 8_3)	{4} {8_3}	2 1	3 1	16/3	16 3	8 2	10	24
(3 3 3 8_3)	{3} {8_3}	3 1	4 1	16/3	16 3	16 2	18	32
(3 5_2 3 5_2 3 5_2)	{3} {5_2}	3 3	6 1	10	20 2	20 12	32	60
(5 5_2 5 5_2)	{5} {5_2}	2 2	4 1	10	30 3	12 12	24	60
(3 8_3 4 8_3)	{3} {4} {8_3}	1 1 2	4 1	6	24 4	8 6 6	20	48
(3 8_3 8_3)	{3} {8_3}	1 2	3 1	24/7	24 7	8 6	14	36
$(3\ 4\ 3\ 4)_2$	{3} {4}	2 2	4 3/2	3	6 7/2	4 3	7	12
$(3\ 6\ 3\ 6)_2$	{3} {6}	2 2	4 3/2	4	12 3	8 4	12	24
$(4\ 6\ 4\ 6)_2$	{4} {6}	2 2	4 3/2	3	12 4	6 4	10	24
$(3\ 10\ 3\ 10)_2$	{3} {10}	2 2	4 3/2	60/11	30 11/2	20 6	26	60
$(5\ 10\ 5\ 10)_2$	{5} {10}	2 2	4 3/2	20	30 3/2	12 6	18	60
$(3\ 8\ 3\ 8)_2$	{3} {4} {8}	1 1 2	4 3/2	6	24 4	8 6 6	20	48
$(4\ 8\ 4\ 8)_2$	{4} {8}	2 2	4 3/2	8	24 3	12 6	18	48
$(4\ 10\ 4\ 10)_2$	{4} {10}	2 2	4 3/2	10	60 6	30 12	42	120
$(3\ 10\ 5\ 10)_2$	{3} {5} {10}	1 1 2	4 3/2	60/7	60 7	20 12 12	44	120
$(3\ 4\ 4\ 4)_2$	{3} {4}	1 3	4 3/2	24/7	24 7	8 18	26	48
(5 5_2 5 5_2 5 5_2)	{5} {5_2}	3 3	6 2	5/2	20 8	12 12	24	60
$(6_2\ 6_2\ 6_2\ 6_2\ 6_2\ 6_2)_2$	{6_2}	6	6 2	2	20 10	20	20	60
(5_2 10 10)	{5_2} {10}	1 2	3 1	20	60 3	12 12	24	90
(5_2 6 6)	{5_2} {6}	1 2	3 1	60/7	60 7	12 20	32	90

6.6 OTHER POSSIBILITIES

In the Introduction to this Chapter the number of 53 has been mentioned as the number of UH1's, which, beside prisms and antiprisms, are presently known. Only a relatively small part of this series has been mentioned in this chapter. A complete survey can be found in the book of Wenninger. However, what is complete? Among the 53 UH1's some have been known for centuries, but other ones have been discovered only relatively short ago! In contrast to the Platonic, Poinsot and Archimedean solids, a definite proof for the number of higher-order uniform polyhedra has not been given as yet. The search is still continuing!

7

MORE THAN THREE DIMENSIONS

7.1 INTRODUCTION

When we considered the various polyhedra, we started off with looking at the polygons out of which they are composed. From these t wo-dimensional polygons we jumped into the third dimension to arrive at the polyhedra. A further jump will lead us into the fourth dimension. The question is now, whether in the fourth dimension solids exist, which are bounded by three-dimensional polyhedra? As a matter of fact, this will indeed be the case, though their “existence” is much more difficult to visualize.

These “polytopes”, composed of three-dimensional solids, have been studied and analysed in detail. When looking at them, we are in a completely different world! In the final chapter of this book, the polytopes will be mentioned only very briefly; a more elaborate treatment would be beyond its scope. Much more information can be gathered from Coxeter's book “Regular Polytopes”.

7.2 THE EIGHT-CELL OR HYPER-CUBE

When we look back to the regular, Platonic solids, then we see how the cube, as a three-dimensional regular polyhedron can be thought to originate from a point (see Figure 7.1):

- in the zero dimension: a point;
- shift this point in the first dimension: a line segment;
- shift the line segment into the second dimension: a square;
- shift the square into the third dimension: a cube.

The question is now: how far can we continue with this procedure? Can we shift the cube into the fourth dimension so as to obtain a regular four-dimensional solid? How would it look like and what would be its properties? Our power of imagination now, unfortunately, fails, and we are left with a strict analytical treatment to define the transition from a 3- into a 4-dimensional object, in analogy to the one from 1 to 2 and from 2 to 3 dimensions.

We then see that with the one-dimensional line segment, shifting of the two ends results in two new corners and in two new edges, so that the resulting twodimensional figure is enclosed by the two new edges and by the original as well as by the shifted line segment, so by four edges.

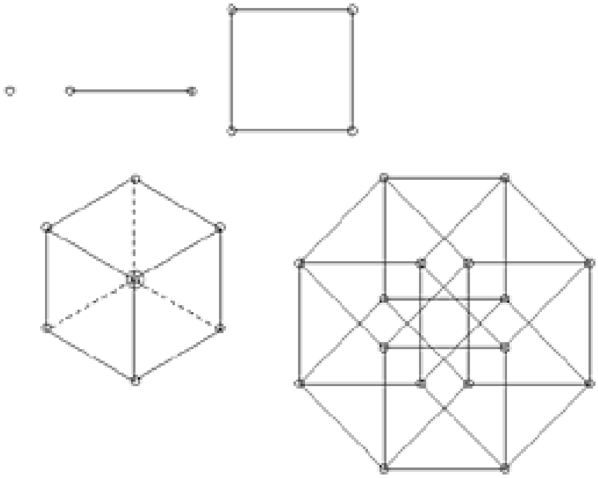


Figure 7.1 From point to 8-cell

Shifting of the square into the third dimension produces four new shifted edges plus four new edges formed by the line segments which connect the original corners with the shifted corners, so in total E = 12. The number of faces is given by a. the original square, b. the shifted square, and c. four squares formed by the shift in the third dimension, so in total F = 6. The number of vertices is twice that of the original square: F = 8.

We now shift a cube in the fourth dimension. Its 8 vertices form another set of 8 vertices, so we already know that the four-dimensional solid has V = 16 vertices. Each vertex produces during shifting a new edge, thus the total number of edges is twice the old one plus the original number of vertices (8), or E = 32. Moreover, upon shifting, each edge creates a new face, and therefore the number of faces becomes twice the original number ($2 \cdot 6 = 12$) plus the original number of edges (= 12), so F = 24.

Finally, hardest to imagine: each square face is shifted to form a cube, just as it happened when the cube was formed. The new solid is now bounded by the original cube, the shifted cube, and the six cubes created by the shift of the faces, in total 8 cubes. The four-dimensional solid is, therefore, called an “eight-cell”; it is bounded by C = 8 cells (cubes), it has V = 16 vertices, F = 24 faces and E = 32 edges.

It appears that, for this solid, V + F = E + C (16 + 24 = 32 +8), an analogue of Euler's law for 3-dimensional solids.

7.3 THE FIVE-CELL

A second polytope that can relatively easily be imagined, is bounded by tetrahedra. We again start off with a point (Figure 7.2) and extend this to a line segment. Now we choose, in the second dimension, a point in such a way that an equilateral triangle is formed. Next we raise a perpendicular line on the plane of the triangle, on which we situate the fourth point, forming a regular tetrahedron with the triangle. The final step is in the fourth dimension, namely via a line perpendicular to the space of the tetrahedron, on which we locate the fifth point. The result is a polytope bounded by 5 tetrahedra, and containing 10 faces, 10 edges and 5 vertices, which can be shown in the same way as with the hypercube. We again find: V + F = E + C .

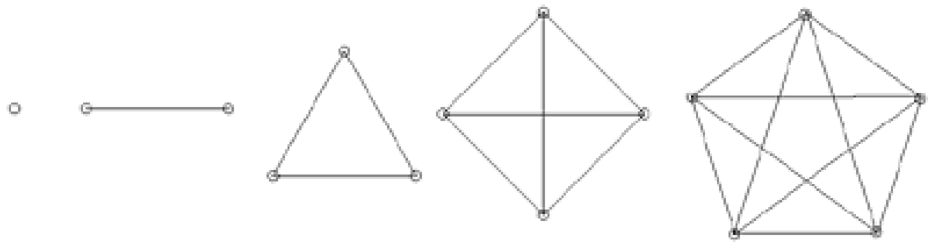


Figure 7.2 From point to five-cell

7.4 THE SIXTEEN-CELL

Finally a third polytope: We construct (Figure 7.3) from a line segment again a square, but in this case not by shifting the line segment, but by choosing two points at its opposite sides, which form a square with the other points. The original line segments thus does not form an edge of the square. Then we build, starting from the square, a double pyramid: a regular octahedron (the square again disappears). The polytope is now formed by situating two points in the fourth dimension and connecting these with each if the twelve vertices of the octahedron.
We can notice that the number of edges, $E = 2 \cdot 12 = 24$, and the number of vertices, $V = 2 + 6 = 8$. Each of the 12 edges of the octahedron forms, together with each of the new vertices, a new triangular face; together with the original 8 faces, $F = 2 \cdot 12 + 8 = 32$. The number of cells (tetrahedra) is twice the number of original faces, so $C = 16$. The initial octahedron again disappears. Here again: V + F = E + C (= 40). It can be shown that this modification of Euler's law for polytopes is generally valid.

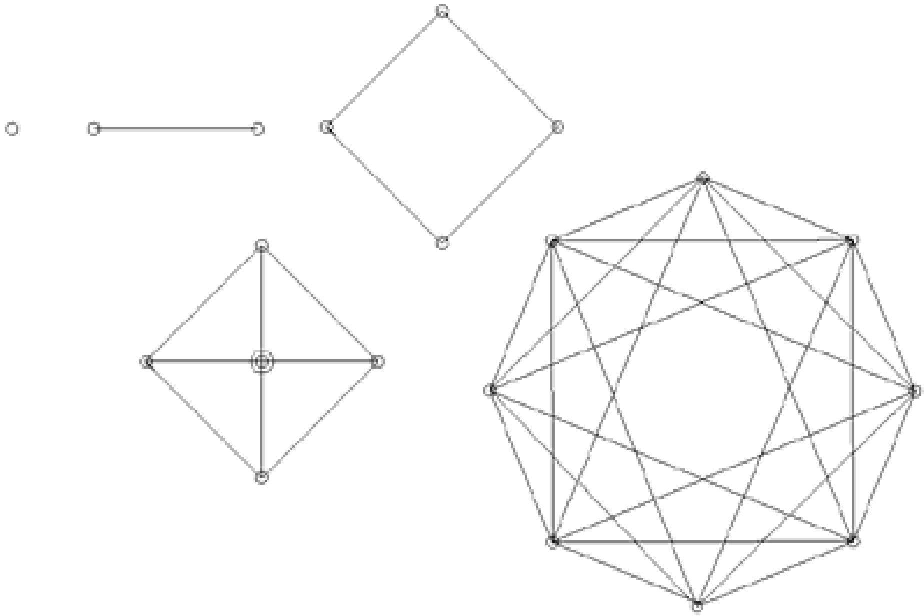


Figure 7.3 From point to 16-cell

7.5 OTHER POLYTOPES

The question is now whether next to the three simplest ones, other regular polytopes exist. An analysis, analogous to the one carried out for Platonic solids, is quite simple, but much more elaborate; therefore, we only look at its results. Three other ones appear to exist, namely the 24-cell, the 120-cell and the 600-cell. The first of these is bounded by octahedra, the second by dodecahedra, the third by tetrahedra.

The most important properties of the six regular polytopes are summarized in the Table below. Their notation is analogous to the one for the Platonic solids: the first number indicates the type of polygon by which a cell is bounded; the second shows how many of these faces meet in a vertex of a cell; the third is the number of cells joining together at a vertex of the polytope.

polytope	cell	notation	V	E	F	C
5-cell	{3 ,3}	{3 ,3,3}	5	10	10	5
8-cell	{4 ,3}	{4 ,3,3}	16	32	24	8
16-cell	{3 ,3}	{3 ,3,4}	8	24	32	16
24-cell	{3 ,4}	{3 ,4,3}	24	96	96	24
120-cell	{5 ,3}	{5 ,3,3}	600	1200	720	120
600-cell	{3 ,3}	{3 ,3,5}	120	720	1200	600

The table shows a remarkable reciprocity between the 8-cell and the 16-cell, and between the 120- and the 600-cell; these form pairs of dually related polytopes, just as with Platonic solids between cube and octahedron and between dodecahedron and icosahedron. For polytopes the numbers of vertices are exchanged with those of cells, as well as the numbers of edges with those of faces. The 5-cell and the 24-cell both appear to be identical to their own duals, just as the Platonic tetrahedron.

In Chapter 5 we have seen that regular polyhedra can also be built up from higher-order polygons or with higher-order polyhedral angles; the four Poinsot-solids appeared to provide an extension of the five Platonic ones. It seems logical to suppose that a similar situation is found in the fourth dimension. Indeed polytopes can originate from stellation, either in their faces or in vertices. It appears that there are ten of them! Their notations are:

$$\{5_2,5,3\},\ \{3,5,5_2\},\ \{5,5_2,5\},\ (5_2,3,5\},\ \{5,3,5_2\},\ \{5_2,5,5_2\},\ \{5,5_2,3\},$$
$$\{3,5_2,5\},\ \{5_2,3,3\},\ \{3,3,5_2\}.$$

Of course, the possibilities for interesting polytopes are certainly not exhausted by the series mentioned so far. Analogous to the series of 13 Archimedean solids and the large number of higher-order half-regular polyhedra, numerous polytopes can be constructed by joining regular but different regular solids, by various types of truncation, by stellation etc. When we also consider their inversions into dually related polytopes, we find ourselves in a world of numerous beautiful, but, unfortunately, unimaginable four-dimensional figures.

Representations of polytopes on a sheet of paper are not satisfactory: in the first place because we miss the power of imagining four-dimensional objects; in the second place a reduction of four to two dimensions means a similar reduction as the representation of, e.g., a stellated dodecahedron on a straight line! Three-dimensional figures, built-up from threads, can of course be made as projections of polytopes on 3-space. The second drawback is then circumvented, but the first remains!

7.6 HIGHER DIMENSIONS

The brief excursion in the fourth dimension may have raised the question: can we go into higher dimensions? Intuitively the idea arises that the recipes for the first three polytopes can be extended further. This is indeed the case for an unlimited number of dimensions.

The series, starting off with line segment, cube and 8-cell, is called the series of “measure polytopes”. The members of the second family: line segment, triangle, tetrahedron and five-cell have the general name “simplex”. The third series, the “cross-polytopes” contains solids analogous to the octahedron.

It can be shown that these series form the only regular polytopes in 5- and higher dimensions; the families of polytopes thus become rather dull!

Far from dull are, however, are the higher-order half-regular polytopes, which are, as in the fourth dimension, abundantly present in higher dimensions. More information can be found in Coxeter, “Regular Polytopes”.

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