# 5

# REGULARITY WITH STARS (POINSOT-SOLIDS)

## 5.1 INTRODUCTION

The description of regular and semi-regular polyhedra could have been finished with the previous Chapter if we would stick to the conventional definition of a polygon in which the sides of the polygon are not allowed to intersect between its corners.

It is, however, possible to drop this requirement; then the number of regular polygons, and also the number of regular and semi-regular polyhedra, increases considerably.

The pentagram, shown in Figure 5.1, can be considered as a regular polygon with corners A, B, C, D and E, since its sides AB, BC, CD, DE and EA are equal, as well as the angles at its corners. The points of intersection of these sides cannot be considered as corners; their function is analogous to the points of intersection of the (elongated) sides of a "normal" polygon.

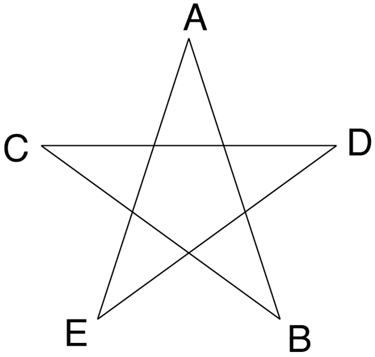

Figure 5.1 Pentagram $\{5_2\}$

This polygon and similar ones are called "higher-order" or "higher-density" polygons. When, in a regular positioning of the corners, we connect a corner with the next one to obtain the edges, we do not take the adjacent corner but, in the example mentioned, the second one. In this way the star-shaped pentagon, the pentagram, is born as a second-order pentagon, indicated by $\{5_2\}$.

How big are the angles in such a higher-order polygon? Starting from A, the edges run to B and to E, so at both sides a corner is being skipped. The angle at A, therefore, comprises two less segments of the circum-circle, and thus it is not:

$$\frac{1}{2}\cdot\frac{5-2}{5}\cdot 360^\circ = 108^\circ \quad \text{but: } \frac{1}{2}\cdot\frac{5-2\times 2}{5}\cdot 360^\circ = 36^\circ.$$

In general, the angle of an n-gon of the order a, $\{n_a\}$ equals $(n - 2a)/n\cdot 180^\circ$, and the sum of the n angles is $(n - 2a)\cdot 180^\circ$.

Other examples of higher-order polygons are (see Figure 5.2) the $\{8_3\}$ and the $\{10_3\}$.

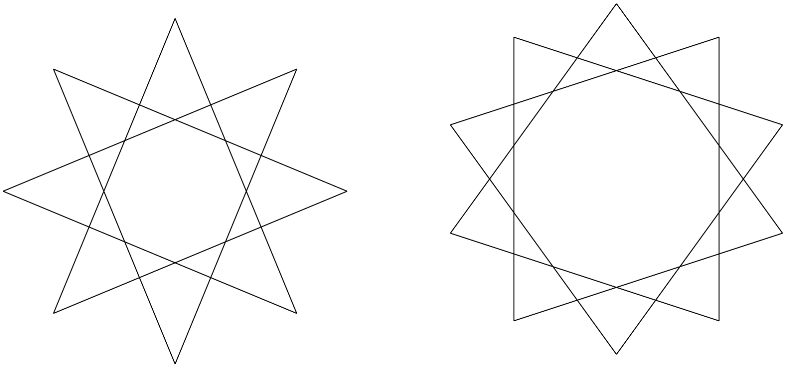

Figure 5.2 Higher-order polygons: $\{8_3\}$ en $\{10_3\}$

In general the order (or the density) of a polygon can be defined as the number of sides which are intersected by a half-line from its midpoint; this line should not pass through one of the corners. Expressed in a different way: When the polygon is “blown-up” from its centre, until all its sides become parts of a circle, the circle is covered as many times as the order of the polygon indicates.

Obviously the existence of higher-order polygons results in an extension of the definition of a polyhedral angle. When we construct a plane through each of the sides of the star-pentagon in such a way that these planes have a common point of intersection, then these planes form together a second-order pentahedral angle or a $5_2$-hedral angle (with functionality 5/2). When the line from this point of intersection perpendicular to the plane of the pentagram, passes through its centre, we have a regular $5_2$-hedral angle.

For regular polyhedral angles of a higher order, the formula in Chapter 1 (form. 1.5) is again valid, which gives the relation between dihedral angles and edges:

$$\cos\varphi = \frac{\cos\alpha - p}{\cos\alpha + 1}$$

in which p = 1 + 2 cos (360°/m), but now we replace m by $m_b$, in which b is the functionality of the polyhedral angle. We thus find for $5_2$, $8_3$, and $10_3$ polyhedral angles, values for p of, respectively $-(\sqrt{5} - 1)/2$, $(1 - \sqrt{2})$, and $(3 - \sqrt{5})/2$.

Further on in this chapter we shall consider the consequences of both of these extensions, of the polygon as well as of the polyhedral angle, with respect to the regular polyhedra. Then also the quantity "order" of polyhedra will be discussed. This will be done in an analogous way as for polygons; when we pass, starting from the mid-point of a polyhedron and travelling outward, c times a face (not in vertices or on edges), the order of a polyhedron is called c. In doing so, the central part of a face with the order a, is counted a times, a part of a face which is separated by one edge from the center part, (a – 1) times, etc.

## 5.2 ANALYSIS

Let the number of faces of a higher-order polyhedron be F, the number of vertices V and the number of edges E. The faces are n-gons of the order a, the vertices form m-hedral angles of the order b.

Independent of the values of a and b, the well-known relations between F, V, E, n and m are valid:

$$F \cdot n = V \cdot m = 2 \cdot E\ .$$

Euler's formula is also valid for higher-order polyhedra, albeit in a modified shape. In the original formula,

$$V + F = E + 2\ ,$$

V has now to be replaced by b·V and F by a·F, in other words: vertices as well as faces are counted as many times as their order indicates. The constant 2 is, moreover, replaced by 2·c (c is the order or the density of the polyhedron). Euler's rule now becomes:

$$b{\cdot}V + a{\cdot}F = E + 2{\cdot}c\ .$$

The proof of this extended formula of Euler will not be given here.

Combination of this formula with the relations given above, results in:

$$F{\cdot}n = V{\cdot}m = 2{\cdot}E = \frac{2 \cdot c}{a/n + b/m - 1/2}$$

For a = b = c = 1 these relations are reduced to those already known from Chapter 1.

When we take these relations as a starting point, we can try out, which combinations result in a possible polyhedron. Let us first consider the polygon $\{5_2\}$. How many possibilities exist to join a number of these polygons into a first-order vertex? As far as the value of the top-angle (36°) concerns, m-values of 3 up to 9 could be considered. To investigate these possibilities, we can write the equations as follows:

$$2\cdot(c/V) = b - (m/n)\cdot(n/2 - a)\ ; \quad F = V\cdot(m/n)\ .$$

For the case under investigation with $n = 5$, $a = 2$, $b = 1$ this becomes:

$$2\cdot(c/V) = 1 - m/10\ , \qquad F = (V/5)\cdot m\ .$$

For various values of m we find several potential possibilities to form a polyhedron. The question is now, whether all these possibilities lead to really existing polyhedra. When we try them out, most of them disappear. This is not surprising, since we have only considered the conditions for joining faces at vertices, such as for the acute angle A of Figure 5.1, but the adjacent corners C and D should also be able to fit into a vertex with other faces.

From further analysis an extra criterion can be derived, namely that the vertices of regular higher-order polyhedra should coincide with those of a first-order, Platonic, polyhedron. Thus V should, for pentagons, be 12 or 20. The same should, of course, hold after dual exchange, so that F can only be 12 or 20. According to this criterion the only remaining possibilities are: $m = 3$ with $c = 7$, and $m = 5$ with $c = 3$. These combinations result in the polyhedra $\{5_2,3\}$ and $\{5_2,5\}$ or, in another notation, $(5_2\ 5_2\ 5_2)$ and $(5_2\ 5_2\ 5_2\ 5_2\ 5_2)$.

Systematic analysis of the other possibilities only results in the dually related pair: $\{3\ , 5_2\}$ and $\{5\ , 5_2\}$ or, respectively, $(3\ 3\ 3\ 3\ 3)_2$ and $(5\ 5\ 5\ 5\ 5)_2$, which can, of course, also be denoted as $((5_2\ 5_2\ 5_2))$ and $((5_2\ 5_2\ 5_2\ 5_2\ 5_2))$. A survey of the parameters of these four solids, together with the related first-order Platonic solids, is given in the table below.

| | n | a | m | b | Z | H | R | c | name |
|---|---|---|---|---|---|---|---|---|---|
| $\{5_2, 3\}$ | 5 | 2 | 3 | 1 | 12 | 20 | 30 | 7 | great stellated dodecahedron |
| $\{5_2, 5\}$ | 5 | 2 | 5 | 1 | 12 | 12 | 30 | 3 | small stellated dodecahedron |
| $\{3, 5_2\}$ | 3 | 1 | 5 | 2 | 20 | 12 | 30 | 7 | great icosahedron |
| $\{5, 5_2\}$ | 5 | 1 | 5 | 2 | 12 | 12 | 30 | 3 | great dodecahedron |
| $\{3, 5\}$ | 3 | 1 | 5 | 1 | 20 | 12 | 30 | 1 | icosahedron |
| $\{5, 3\}$ | 5 | 1 | 3 | 1 | 12 | 20 | 30 | 1 | dodecahedron |

The four higher-order regular polyhedra are called the “Kepler-Poinsot solids”. Kepler described the first two of them; the other two were first discovered by Poinsot (1810), who presented the complete collection of higher-order regular polyhedra as an addition to the five Platonic solids.

## 5.3 THE GREAT STELLATED DODECAHEDRON {$5_2$,3}

Though this polyhedron is, of the four Kepler-Poinsot solids, the most closely related to the dodecahedron {5 ,3}, it can easiest be thought to originate from an icosahedron {3,5} by stellation. When the edges of the R20 are extended into both directions, until they intersect three by three, at each face of the icosahedron three-sided pyramids are built which together form a 20-pointed star (see Figure 5.3). Each of the faces of such a pyramid is part of a pentagram {$5_2$}; the middle part of this pentagram is formed by the intersection of the icosahedron with a plane through five vertices adjacent to a vertex. These 12 pentagons enclose a regular dodecahedron, which is the core of the great stellated dodecahedron.
From the analysis it follows that the order of the {$5_2$,3} is 7: this can also be seen by travelling along a half-line from its centre; we then pass four faces, namely three times a double-counting central part of a pentagram and once an outer part.

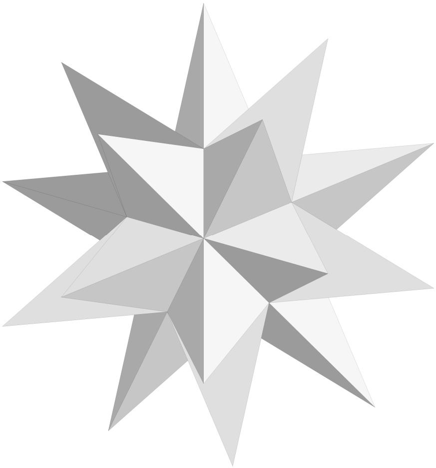

Figure 5.3 Great stellated dodecahedron

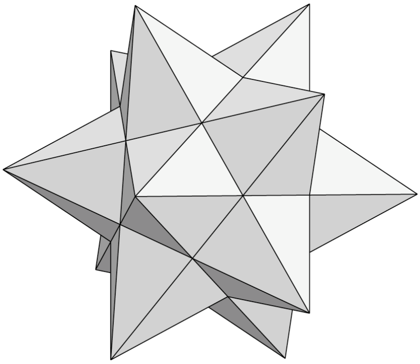

Figure 5.4 Small stellated dodecahedron

## 5.4 THE SMALL STELLATED DODECAHEDRON {$5_2$,5}

In the same way as the {$5_2$,3}, as discussed in the previous section, could be thought to originate from the {3,5} by stellation, the small stellated dodecahedron {$5_2$,5} can be formed from the dodecahedron {5,3}. In this case, however, the planes of their faces coincide: the {$5_2$,5} is created by extending the 12 faces of the {5,3} (see Figure 5.4). The pentagons are thereby extended to pentagrams {$5_2$}, five of which meet in a vertex. The number of vertices equals the number of faces, namely 12, which is a consequence of the fact that m = n, which we, so far, only encountered with the tetrahedron (m = n = 3, V = F = 4).

The order of $\{5_2,5\}$ is 3; starting from its centre we meet two faces, once we pass through the double-counting central part of a pentagram and once through one of the points.

The core of the small stellated dodecahedron is formed by the dodecahedron mentioned before. The vertices coincide with those of an icosahedron.

When we compare the $\{5_2,3\}$ with the $\{5_2,5\}$ on the basis of their common core, the dodecahedron, with both of which they have the planes of their 12 faces in common, the names “great” and “small” are clear: the great one has a $(5\sqrt{5} + 11)/2 = 11.1$ times longer edge then the inscribed dodecahedron, and the small one $(\sqrt{5} + 2) = 4.24$ times greater. The edges of the great and the small ones are, therefore, in the ratio 2.62 : 1.

## 5.5 THE GREAT ICOSAHEDRON $\{3 , 5_2\}$

This solid is narrowly related to the icosahedron $\{3,5\}$ (the same number of vertices, faces and edges), and it can be derived from $\{3,5\}$ in two different ways, namely either via its vertices or via its faces.

Using the vertices, we can inscribe a $\{3,5_2\}$ into a $\{3,5\}$: its edges are the spatial diagonals of $\{3,5\}$ which do not pass through the centre, as indicated in Figure 5.5 for two series of three edges, each forming a face of $\{3,5_2\}$. In each vertex five of the 20 faces meet together in a second-order polyhedral angle

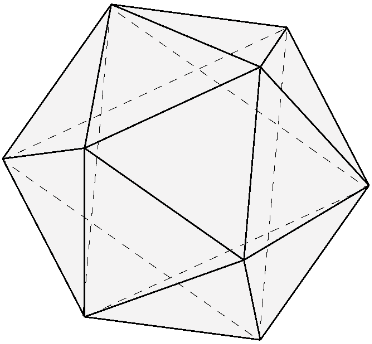

Figure 5.5 Two faces of $\{3 , 5_2\}$ in $\{3 ,5\}$

If we start off from the faces of an icosahedron, then we obtain a circumscribed $\{3,5_2\}$, which has the $\{3,5\}$ as a core. If we take a face of $\{3,5\}$ and intersect it with all other faces (except the opposite one) then we obtain a pattern as indicated in Figure 5.7. This figure contains 18 line segments, which form the intersections with the 18 faces. In this figure we can recognize the compounds of five octahedra and of five tetrahedra. The inner triangle is the face of the original icosahedron, the outer triangle is the face of the great icosahedron. From the figure it is clear that the order of $\{3,5_2\}$ is 7. The only parts of the faces which are visible from the outside, are the nine triangles which are adjacent to the outer circumference.

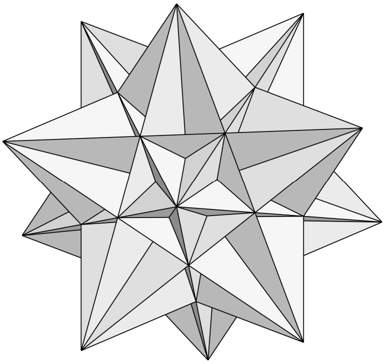

Figure 5.6 Great icosahedron $\{3,5_2\}$

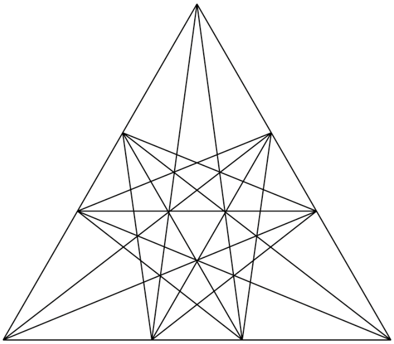

Figure 5.7 Face of $\{3,5_2\}$ with intersections

When we compare $\{3,5_2\}$ with $\{5_2,5\}$, then it appears that not only their vertices, but also their edges can coincide. The great icosahedron can, therefore, be very narrowly enclosed by the small stellated dodecahedron!

## 5.6 THE GREAT DODECAHEDRON $\{5,5_2\}$

A further look at the icosahedron $\{3,5\}$ reveals that it contains 12 $\{5\}$-shaped diagonal faces. These faces form together the great dodecahedron, which, therefore, has the same vertices and edges as the icosahedron.

Another way to create this solid, is by extending the pentagram-shaped faces of the small stellated dodecahedron $\{5_2,5\}$ to pentagons $\{5\}$ with the same corners. In this way the relation with the other three dodecahedra becomes clear; this is demonstrated in Figure 5.8.

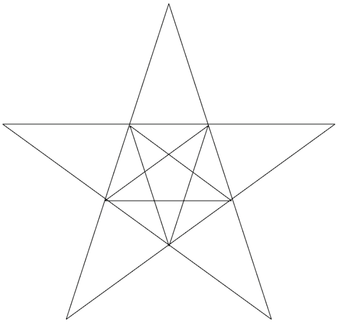

Figure 5.8 Faces of the four dodecahedra

At the inside the pentagon is situated which forms the face of the dodecahedron {5,3}. When we elongate the edges to form a pentagram {$5_2$} we obtain the small stellated dodecahedron {$5_2$,5}. Extension of the pentagram to a pentagon {5} creates the great dodecahedron {5,$5_2$}; extending the edges of this pentagon into a pentagram {$5_2$} leads to the great stellated dodecahedron {$5_2$,3}. We cannot continue: connecting the corners of the largest pentagram to a pentagon results in incoherent faces.

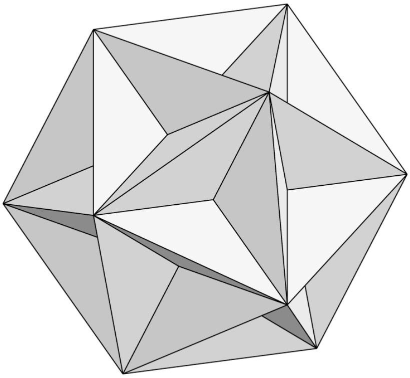

Figure 5.9 Great dodecahedron {5 , $5_2$}

The outer surface of the great dodecahedron is composed of 60 isosceles triangles with angles of 108°, 36° and 36°, which, three by three, form hollow pyramids in the faces of the enveloping icosahedron; the obtuse tops of these pyramids are the vertices of the dodecahedron which forms the core of the great dodecahedron.

## 5.7 NINE PLATONIC SOLIDS

In this chapter we have seen that the extension of the notions “regular polygon” and “regular polyhedral angle” leads to the completion of the number of Platonic solids from 5 to 9. We also saw, though not fully proven, that further extension of this series is not possible.

In view of the narrow relationship of the four new solids with the dodecahedron and the icosahedron, it is not surprising that all ways of fitting the latter into tetrahedron, cube and octahedron, also hold for the Poinsot solids.

Our next step will now be, to investigate whether the extended definitions of regular polygons and polyhedral angles will also lead to new semi-regular solids. This will be dealt with in the following chapter.