5 Numerical quadrature

5.1 Introduction

Determining the physical attributes of a system (for e.g. the volume, the mass, the length) often involves the integral of a function. Situations may arise where an analytic evaluation of the integral may not be possible; in such cases one resorts to numerical quadrature.

As an example we take the production of a spoiler, that is mounted onto the cabin of a truck (figure 5.1). The shape of the spoiler is described by a sine function with a 2π meter period. The spoiler is made out of a flat plate by extrusion. The manufacturer wants to know the width of the plate such that the horizontal dimension of the spoiler will be 80 cm. The answer to that is the arc length of the curve given by

$$\begin{aligned} x(t) &= t \qquad\qquad 0 \le t \le 0.8 \,. \\ y(t) &= \sin t \end{aligned}$$

To determine the arc length we may use the formula

$$l = \int_0^{0.8} \sqrt{1 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{0.8} \sqrt{1 + (\cos t)^2} dt \,.$$

The integral cannot be evaluated in a simple way. In this chapter we shall show how we can use numerical quadrature to determine the required length.

5.2 Simple numerical quadrature formulae

First we shall recall the definition of an integral and after that we shall give a few simple quadrature rules. The approximation error of those rules and the effect of rounding errors will also be considered.

Definition
A *partition* P of $[a,b]$ is a finite number of points x_k where $a = x_0 < x_1 ... < x_n = b$. A corresponding *scattering* T is a set of intermediate points t_k such that $x_{k-1} \le t_k \le x_k$. Let us denote the length of an interval by $h_k = x_k - x_{k-1}$ and the mesh width by $m(P) = \max_{1 \le k \le n} \{h_k\}$. The *Riemannian sum* for a function f continuous on $[a,b]$ is defined by:

$$R(f,P,T) = \sum_{k=1}^{n} f(t_k) h_k \,.$$

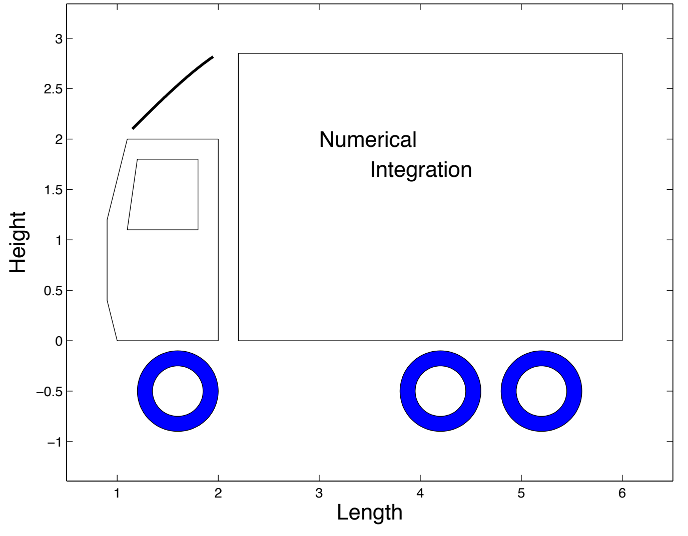

Figure 5.1 Truck with spoiler on the cabin.

Choose a sequence of partitions $P_1, P_2, \ldots$ and corresponding scatterings T_n such that $m(P_n) \to 0$ then

$$R(f, P_n, T_n) \quad \text{converges to a limit} \quad I = \int_a^b f(x)dx \,.$$

Numerical quadrature rules are very similar to the Riemannian sum. The most important difference is the required efficiency of numerical quadrature.

The rectangle rule

Take an equidistant partition such that the nodal points are given by: $x_k = a + kh$, $k = 0, 1, \ldots, n$ with $h = \frac{b-a}{n}$. We take the scattering points equal to the left hand nodal points: $t_k = x_{k-1}$. The corresponding Riemannian sum is given by

$$I_R = h[f(a) + f(a+h) + \ldots + f(b-h)] \,.$$

Theorem 5.2.1 *Let f be differentiable on $[a,b]$. Suppose M_1 is the maximum of $|f'|$ on $[a,b]$. Then:*

$$\left| \int_a^b f(x)dx - I_R \right| \leq \frac{1}{2} M_1 (b-a) h$$

Proof:

First we consider the interval $[x_{k-1}, x_k]$. From Taylor expansion we have:

$$f(x) = f(x_{k-1}) + (x - x_{k-1}) f'(\xi(x)) \quad \text{with} \quad x_{k-1} \leq \xi(x) \leq x_k \,.$$

And from this we obtain:

$$\left| \int_{x_{k-1}}^{x_k} [f(x) - f(x_{k-1})]dx \right| \leq M_1 \int_{x_{k-1}}^{x_k} (x - x_{k-1})dx ,$$

and therefore

$$\left| \int_{x_{k-1}}^{x_k} f(x)dx - hf(x_{k-1}) \right| \leq \frac{1}{2} M_1 h^2 .$$

For the total error we get:

$$\left| \int_a^b f(x)dx - I_R \right| \leq \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f(x)dx - hf(x_{k-1}) \right| \leq \frac{1}{2} M_1 h^2 n = \frac{1}{2} M_1 (b-a)h .$$

⊠

Example 5.2.1 (spoiler)

For the example mentioned in the introduction we want to approximate the length of the flat plate with an error of at most 1 cm. Applying the rectangle rule we must take care that

$$\frac{1}{2} M_1 0.8h \leq 0.01 .$$

The derivative of the integrand $f(t) = \sqrt{1 + (\cos t)^2}$ is given by:

$$f'(t) = \frac{-\cos t \sin t}{\sqrt{1 + (\cos t)^2}} = \frac{-\frac{1}{2} \sin 2t}{\sqrt{1 + (\cos t)^2}} .$$

From this it follows that $|f'| \leq M_1 \leq \frac{1}{2}$ and therefore a step size $h = 0.05$ meets the requirement. The integral is 1.0759 m in 5 digit precision. For $n = 16$ the rectangle rule gives 1.0807, hence the error is, in fact, less than 1 cm.

The accuracy of the rectangle rule is $O(h)$. In the next section we shall present a more accurate method that uses the same amount of work.

Midpoint rule

Take an equidistant partition, such that the nodes are given by $x_k = a + kh$, $k = 0, 1, ..., n$ and $h = \frac{b-a}{n}$. The midpoints $\frac{x_k + x_{k-1}}{2}$ are chosen as scattering points. The corresponding Riemannian sum is given by:

$$I_m = h[f(a + \frac{1}{2}h) + f(a + \frac{3}{2}h) + ... + f(b - \frac{1}{2}h)] .$$

Theorem 5.2.2 *Let f be twice differentiable on $[a,b]$. Suppose M_2 is the maximum of $|f''|$ in $[a,b]$ then:*

$$\left| \int_a^b f(x)dx - I_m \right| \leq \frac{1}{24} M_2 (b-a)h^2 .$$

Proof. Prove this theorem yourself.
(Hint: Expand f about $\frac{x_k+x_{k-1}}{2}$ in a Taylor polynomial of degree 1). ⊠

Example 5.2.2 (spoiler)

In Table 5.1 the errors have been tabulated for different values of the step size h. From

Table 5.1 The error for different values of h.

h	Rectangle rule	Midpoint rule
0.8	– 0.055	– 0.0117
0.4	– 0.0336	– 0.0028
0.2	– 0.0182	– 0.00068
0.1	– 0.0094	– 0.00017
0.05	– 0.0048	– 0.000043

theory we expect that by halving h, the error of the rectangle rule would decrease by a factor of 2 and that of the midpoint rule by a factor of 4. The results are in agreement with these expectations.

Measuring and rounding errors can play an important role also in numerical quadrature. Let us assume that the function values have been perturbed by an error ε:

$$\hat{f}(x) = f(x) + \varepsilon(x) .$$

Note that

$$\left| \int_a^b \left(f(x) - \hat{f}(x)\right) dx \right| \le \int_a^b |f(x) - \hat{f}(x)| dx \le \varepsilon_{\max}(b-a).$$

Approximating the integral with the rectangle rule we get:

$$\left| \int_a^b f(x)dx - h\sum_{i=0}^{n-1} \hat{f}(x_i) \right| \le \frac{1}{2} M_1 (b-a)h + h\sum_{k=0}^{n-1} \varepsilon(x_k) .$$

Assuming $|\varepsilon(x)| \le \varepsilon_{\max}$ we obtain for the total error:

$$\left| \int_a^b f(x)dx - \hat{I}_R \right| \le (\frac{1}{2} M_1 h + \varepsilon_{\max})(b-a) .$$

Note that it is useless to take h any smaller than $\frac{2\varepsilon_{\max}}{M_1}$.

Example 5.2.3 (spoiler)

We did the computation of the length of the flat plate again under the assumption of a small manufacturing error. As a consequence of that, the integrand contains a rounding error $\varepsilon(x) = 10^{-3}$. In Figure 5.2 we see the effect of the rounding errors: the total error is still larger than 0.8×10^{-3}. It serves no purpose to take the step size smaller than

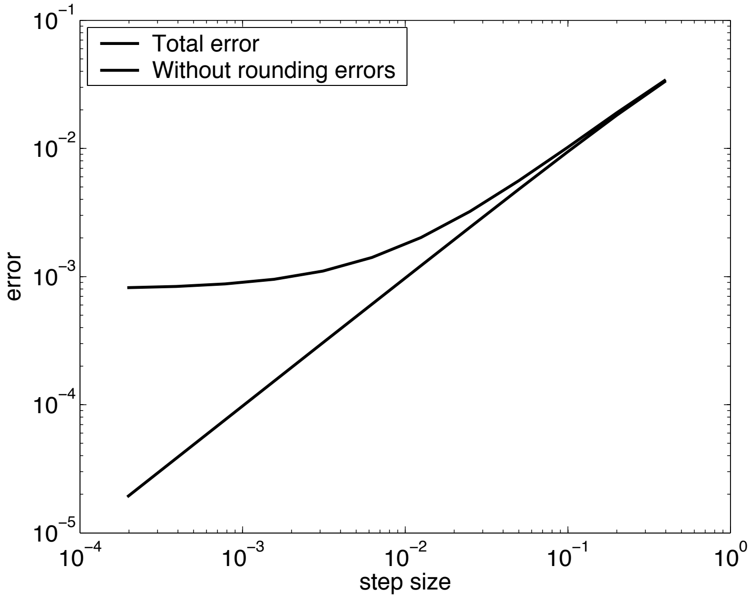

Figure 5.2 The error in the determination of the length of the flat plate.

$4\varepsilon_{\max} = 4 \cdot 10^{-3}$.

Finally we may distinguish between a well posed and an ill posed problem in numerical determination of an integral I. Let us assume that the error is bounded by the inequality:

$$|f(x) - \hat{f}(x)| \leq |f(x)|\varepsilon \ .$$

An upper bound for the relative error in the answer is given by

$$\frac{|\int_a^b f(x)dx - \int_a^b \hat{f}(x)dx|}{\left|\int_a^b f(x)dx\right|} \leq \frac{\int_a^b |f(x)|dx}{\left|\int_a^b f(x)dx\right|} \cdot \varepsilon \ .$$

Define $K_I = \frac{\int_a^b |f(x)|dx}{|\int_a^b f(x)dx|}$ as the condition number of the integral I. If $K_I \gg 1$ then the determination of I is an ill posed problem.

Example 5.2.4 (profits/losses)

The profits or losses per day of a car manufacturer depend on the season. In winter fewer cars are sold than in summer and therefore costs exceed income and the manufacturer suffers a loss. We assume the following profits formula (in billions of \$) in the first quarter of the year:

$$w_{\text{spring}}(t) = 0.01 + \sin(\pi t - \frac{\pi}{2}) \ , \ t \in [0,1]$$

and in the third quarter:

$$w_{\text{fall}}(t) = 0.01 + \sin(\pi t + \frac{\pi}{2}) \ , \ t \in [0,1] \ .$$

The total profits in the first quarter (W_{spring}) equal:

$$W_{\text{spring}} = \int_0^1 w_{\text{spring}}(t)dt \ .$$

In both quarters profits amount to 0.01 (is 10 million \$). Since

$$\int_0^1 |w_{\text{spring}}(t)|dt \simeq 0.63$$

we have $K_I = 63$ and the determination of the integral is an ill posed problem. If we determine both integrals by the rectangle rule we obtain with $n = 50$:

$$\begin{aligned} W_{\text{spring}} &= -0.01 \ , \\ W_{\text{fall}} &= 0.03 \ . \end{aligned}$$

The small step size $h = 0.02$ is insufficiently small to determine the profits correctly.

5.3 Newton-Cotes formulae

In this section we shall describe general quadrature rules. We start with the Trapezoidal rule and we establish the connection between numerical quadrature and interpolation. Subsequently we shall discuss quadrature rules based on higher order interpolation. We will consider Newton-Cotes formulae as a particular case.

The Trapezoidal rule

Let a and b be points in $\mathbb{R}$, $a < b$ and let f be a function, continuous on $[a,b]$. Let p be the linear interpolation polynomial of f based on the nodes a and b. (See Section 2.2):

$$p(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) \ .$$

We take $\int_a^b p(x)dx$ as an approximation of $\int_a^b f(x)dx$:

$$\int_a^b p(x)dx = \frac{b-a}{2}(f(a)+f(b)) \; . \tag{5.1}$$

This approximation is called the *Trapezoidal rule*, since (5.1) equals the area of the trapezium with vertices $(a,0)$, $(b,0)$, $(b,f(b))$ and $(a,f(a))$.

Remainder term

Linear interpolation through $(a,f(a))$ and $(b,f(b))$ has truncation error

$$f(x)-p(x) = \frac{1}{2}(x-a)(x-b)f''(\xi(x)) \; , \tag{5.2}$$

where we write $\xi(x)$ instead of ξ to emphasize the dependence of ξ on x. We integrate both sides of Equation (5.2) to obtain:

$$\int_a^b f(x)dx - \frac{b-a}{2}(f(b)+f(a)) = \frac{1}{2}\int_a^b (x-a)(x-b)f''(\xi(x))dx \; . \tag{5.3}$$

This remainder term may be estimated with the following theorem:

Theorem 5.3.1 *Let $f \in C^2[a,b]$. Then there exists an $\eta \in [a,b]$ such that*

$$\int_a^b f(x)dx - \frac{b-a}{2}(f(a)+f(b)) = \frac{-(b-a)^3}{12} f''(\eta) \; .$$

Proof:

Let $m = \min_{x\in[a,b]} f''(x)$ and $M = \max_{x\in[a,b]} f''(x)$. Since $(x-a)(b-x) \geq 0$ for x in $[a,b]$:

$$\begin{aligned} m\int_a^b (x-a)(b-x)dx &\leq \int_a^b (x-a)(b-x)f''(\xi(x))dx \\ &\leq M\int_a^b (x-a)(b-x)dx \; . \end{aligned}$$

Hence there exists a $\mu \in (m,M)$ such that

$$\int_a^b (x-a)(b-x)f''(\xi(x))dx = \mu\int_a^b (x-a)(b-x)dx = \frac{\mu(b-a)^3}{6} \; .$$

Because f'' is continuous, there exists an $\eta \in [a,b]$ such that $\mu = f''(\eta)$. Substitution into (5.3) completes the proof. ⊠

If we use the Trapezoidal rule to approximate an integral the error is usually larger than the required accuracy allows. A more accurate answer can be obtained by the *composite*

Trapezoidal rule. In that case we subdivide the interval into n parts of size $h = \frac{b-a}{n}$. In each subinterval $[x_{k-1}, x_k]$ with $x_k = a + kh$ we apply the Trapezoidal rule:

$$\int_{x_{k-1}}^{x_k} f(x)dx = \frac{h}{2}[f(x_k) + f(x_{k-1})] - \frac{h^3}{12} f''(\eta_k) ,$$

with $x_{k-1} \leq \xi_k \leq x_k$. Hence

$$\int_a^b f(x)dx = h[\frac{1}{2}f(x_0) + f(x_1) + ... + f(x_{n-1}) + \frac{1}{2}f(x_n)] - \frac{h^3}{12}\sum_{k=1}^{n} f''(\xi_k) . \tag{5.4}$$

Since for a continuous function f'':

$$\min_k f''(\xi_k) \leq \frac{1}{n}\sum_{k=1}^{n} f''(\xi_k) \leq \max_k f''(\xi_k)$$

there exists a $\xi \in [a,b]$ such that

$$\sum_{k=1}^{n} f''(\xi_k) = nf''(\xi) .$$

Substitution into (5.4) gives

$$\int_a^b f(x)dx = I_T(h) - \frac{b-a}{12} h^2 f''(\xi) ,$$

in which $I_T(h) = h[\frac{1}{2}f(x_0) + f(x_1) + ... + \frac{1}{2}f(x_n)]$.

Remarks

1. The remainder term of the composite Trapezoidal rule is of $O(h^2)$, whereas the Rectangle rule has a remainder term of $O(h)$. Both methods require the same amount of work. Hence the Trapezoidal rule is clearly preferred.

2. The composite Trapezoidal rule may also be interpreted as follows. Suppose s is the linear spline approximation of f in $a = x_0, ..., x_n = b$. Then

$$I_T(h) = \int_a^b s(x)dx .$$

Example 5.3.1 (profits/losses)

We also have approximated the profits from the example in Section 5.2 by the Trapezoidal rule. For $n = 50$ we obtain:

$$W_{\text{spring}} = 0.01 \quad \text{and} \quad W_{\text{fall}} = 0.01 .$$

Note that these results are exact. However, if we perturb the integrand: $\hat{w}_{\text{spring}} = w_{\text{spring}} + \varepsilon$, then $\frac{|\hat{W}_{\text{spring}}|}{|W_{\text{spring}}|} \leq 63\varepsilon$. So the problem remains ill posed with respect to perturbations.

General quadrature formulae

We saw that linear interpolation leads to the Trapezoidal rule. If we use other interpolation formulae we get different quadrature rules.

Let $x_0,...,x_m \in [a,b]$. Let p be the Lagrangian interpolation polynomial of f at the nodal points $x_0,...,x_m$. We take $\int_a^b p(x)dx$ as an approximation to $\int_a^b f(x)dx$. Since $p(x) = \sum_{k=0}^{m} f(x_k)L_{km}(x)$, we have

$$\int_a^b p(x)dx = \sum_{k=0}^{m} w_k f(x_k) \ ,$$

with coefficients

$$w_k = \int_a^b L_{km}(x)dx \ ,$$

that apparently do not depend on f. We call $\sum_{k=0}^{m} w_k f(x_k)$ the *Newton-Cotes* quadrature formula for $\int_a^b f(x)dx$ based on interpolation. The x_k are called the nodal points and w_k the *weights* of the quadrature formula. Newton-Cotes formulae of higher order are seldom used; negative weights will occur and that is less desirable.

Theorem 5.3.2 *If f is a polynomial of degree at most m, then every $m+1$-point quadrature formula based on interpolation is exact for f:*

$$\int_a^b f(x)dx = \sum_{k=0}^{m} w_k f(x_k) \ .$$

Proof:

From Theorem 2.3.1 it follows that f coincides with its Lagrangian interpolation polynomial on $x_0,...,x_m$ and hence

$$\int_a^b f(x)dx = \int_a^b p(x)dx \ . \qquad \boxtimes$$

Conversely:

Theorem 5.3.3 *If $x_0,..,x_m$ and $w_0,...,w_m$ are given numbers and if*

$$\int_a^b p(x)dx = \sum_{k=0}^{m} w_k p(x_k) \qquad (5.5)$$

holds for all polynomials p of degree $\leq m$*, then* $\sum_{k=0}^{m} w_k f(x_k)$ *is the quadrature formula based on interpolation for* $\int_a^b f(x)dx$*.*

Proof:
Let p be the interpolation polynomial of f on the nodes $x_0, \dots, x_m$. Is it true that

$$\int_a^b p(x)dx = \sum_{k=0}^{m} w_k f(x_k)?$$

It is, since

$$\int_a^b p(x)dx = \sum_{k=0}^{m} w_k p(x_k) = \sum_{k=0}^{m} w_k f(x_k) \ .$$

Furthermore it can be proved that the weights $w_0, \dots, w_m$ are uniquely determined by (5.5).
⊠

We also can use this theorem to calculate the weights w_k when the nodes $x_0, \dots, x_m$ are known. Take $p(x)$ consecutively $1, x, \dots, x^m$. This provides a system of $m+1$ equations in $m+1$ unknowns w_k. It may be shown, that this system is uniquely solvable.

Midpoint rule
If we take the zeroth order interpolation of f at the point $m = \frac{1}{2}(a+b)$ we find the quadrature formula $\int_a^b f(x)dx$:

$$(b-a)f(m) \ .$$

We call this the *Midpoint rule*. This rule is more accurate than you might expect at first sight. If $f \in C^2[a,b]$ then

$$f(x) = f(m) + (x-m)f'(m) + \frac{1}{2}(x-m)^2 f''(\xi(x))$$

and

$$\int_a^b f(x)dx = (b-a)f(m) + \frac{1}{2}\int_a^b (x-m)^2 f''(\xi(x))dx \ .$$

Since $(x-m)^2 \geq 0$ we can derive

$$\int_a^b f(x)dx - (b-a)f(m) = \frac{(b-a)^3}{24} f''(\eta) \ .$$

Example 5.3.2 (profits/losses)

When we calculate profits or losses with the composite Midpoint rule

$$h[f(a+\frac{1}{2}h) + \dots + f(b-\frac{1}{2}h)] \ ,$$

we obtain, with $n = 50$,

$$W_{\text{spring}} = 0.01 \quad \text{and} \quad W_{\text{fall}} = 0.01 \, .$$

Newton-Cotes formulae
Let $x_0, \ldots, x_m$ be equidistant in $[a,b]$ with $x_0 = a$ and $x_m = b$. Quadrature formulae generated by integration of the m-th order interpolation polynomial p_m of f are called *Newton-Cotes* formulae.

For the remainder terms of these formulae we have the following theorem:

Theorem 5.3.4 *The remainder term R_m of the approximation of $\int_a^b f(x)dx$ by Newton Cotes quadrature formulae satisfies:*

if m is even and $f \in C^{m+2}[a,b]$:

$$\begin{aligned} R_m &= C_m \left(\frac{b-a}{m} \right)^{m+3} f^{(m+2)}(\xi) \quad \textit{with} \\ C_m &= \frac{1}{(m+2)!} \int_0^m t^2(t-1)\ldots(t-m)dt \, ; \end{aligned}$$

if m is odd and $f \in C^{m+1}[a,b]$:

$$\begin{aligned} R_m &= D_m \left(\frac{b-a}{m} \right)^{m+2} f^{(m+1)}(\xi) \quad \textit{with} \\ D_m &= \frac{1}{(m+1)!} \int_0^m t(t-1)\ldots(t-m)dt \, . \end{aligned}$$

Simpson's rule
Now we choose as nodal points $x_0 = a$, $x_1 = \frac{1}{2}(a+b)$ and $x_2 = b$. The corresponding quadrature rule becomes

$$\frac{b-a}{6}[f(x_0) + 4f(x_1) + f(x_2)] \, .$$

It is not difficult to see that Simpson's rule is exact for the polynomials $f(x) = 1$, $f(x) = x - x_1$, $f(x) = (x-x_1)^2$ and $f(x) = (x-x_1)^3$.

5.4 Gauss' formulae*

We derived quadrature rules $\sum_{k=0}^{m} w_k f(x_k)$ by choosing the nodes x_k beforehand and subsequently determining w_k such that the rule is exact for polynomials of degree $\leq m$. Alternatively you could determine w_k *and* x_k in such a way that polynomials of a degree as high as possible will be integrated exactly. You may hope that this would also lead to better results for arbitrary functions. The requirement that the polynomials $1, x, \ldots, x^{2m+1}$ are integrated exactly generates a system of $2m+2$ nonlinear equations in $2m+2$ unknowns w_k and x_k. It can be shown that this system has a solution. The resulting quadrature formulae are called *Gauss' formulae*. The $m+1$ point Gauss' formula has remainder term:

$$R = \frac{(b-a)^{2m+3}((m+1)!)^4}{(2m+3)((2m+2)!)^3} f^{(2m+2)}(\xi) \, .$$

Example 5.4.1 (2 point Gauss' formula)

Suppose we want to determine c_0, c_1, x_0 and x_1 such that the quadrature formula

$$\int_{-1}^{1} f(x)dx \approx c_0 f(x_0) + c_1 f(x_1)$$

gives an exact result when f is a polynomial of degree at most 3. We choose f consecutively $1, x, x^2$ and x^3, hence c_0, c_1, x_0 and x_1 must satisfy

$$\begin{aligned}
f(x) = 1 \Rightarrow \qquad & c_0 + c_1 &= \int_{-1}^{1} 1dx = 2 \, , \\
f(x) = x \Rightarrow \qquad & c_0 x_0 + c_1 x_1 &= \int_{-1}^{1} xdx = 0 \, , \\
f(x) = x^2 \Rightarrow \qquad & c_0 x_0^2 + c_1 x_1^2 &= \int_{-1}^{1} x^2 dx = \frac{2}{3} \, , \\
f(x) = x^3 \Rightarrow \qquad & c_0 x_0^3 + c_1 x_1^3 &= \int_{-1}^{1} x^3 dx = 0 \, .
\end{aligned}$$

It is simple to see, that the solution of this system is given by

$$c_0 = 1 \, , \quad c_1 = 1 \, , \quad x_0 = \frac{-\sqrt{3}}{3} \text{ and } x_1 = \frac{\sqrt{3}}{3} \, ,$$

hence the quadrature formula becomes

$$\int_{-1}^{1} f(x)dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \, .$$

Example 5.4.2 (comparison of methods)

In Table 5.2 the remainder terms have been tabulated, for the calculation of $\int_0^{\pi} \sin x dx$ with the same number of points in Trapezoidal rule, Simpson's rule and the 5-point Gauss' formula. It is remarkable that 5-point Gauss (simple) already gives such a good result, and also that halving the interval increases the accuracy so rapidly in 5-point Gauss.

Table 5.2 Comparison of different quadrature methods.

method	number times composite	number points	error
Trapezoidal	4	5	$1.04 \cdot 10^{-1}$
Simpson	2	5	$4.56 \cdot 10^{-3}$
5p. Gauss	1	5	$1.11 \cdot 10^{-7}$
Trapezoidal	8	9	$2.58 \cdot 10^{-2}$
Simpson	4	9	$2.69 \cdot 10^{-4}$
5 p. Gauss	2	10	$1.1 \cdot 10^{-10}$
Trapezoidal	16	17	$6.43 \cdot 10^{-3}$
Simpson	8	17	$1.66 \cdot 10^{-5}$
5 p. Gauss	4	20	$1.1 \cdot 10^{-13}$

5.5 Summary

In this chapter the following subjects have been discussed:

- Numerical quadrature
- Rectangle rule
- Midpoint rule
- Composite rules
- Newton Cotes formulae
- Trapezoidal rule
- Simpson's rule
- Gauss' formulae

5.6 Exercises

1. We want to determine the following integral:

$$\int_{-1}^{1} [(10x)^3 + 0.001]dx.$$

   (a) The relative rounding error in the function values is less than ε. Determine the relative error in the integral due to the rounding errors.

   (b) We use the composite midpoint rule as numerical quadrature method and $\varepsilon = 4 \times 10^{-8}$. Give a reasonable value for the step size h.

2. Determine $\int_{0.5}^{1} x^4 dx$ with the Trapezoidal rule. Estimate the error and compare this with the real error. Also compute the integral with the composite Trapezoidal rule using step size $h = 0.25$. Estimate the error with Richardson's error estimate.