Chapter 3

The sound of a cup of coffee

3.1 Sound: vibrating air

In the previous chapter we dealt with phenomena of light. This chapter sheds some light on sound, which also plays an important part in our lives. In contrast to light, sound can not propagate through vacuum. It needs a medium, more precisely sound is small amplitude vibrations inside a medium. This might be water, air but equally well a solid. Nevertheless, we are most used to sound in air.

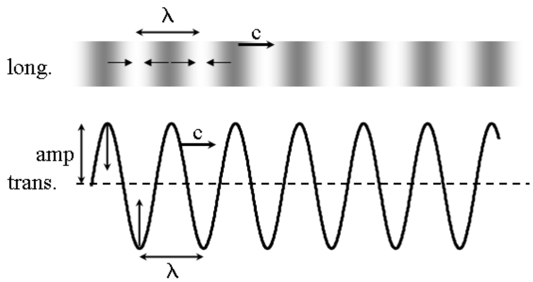


Figure 3.1 Different wave types: (top) longitudinal waves, (bottom) transversal waves.

Sound can be considered as small amplitude oscillations of the density of, say, air. These oscillations go hand in hand with variation in pressure. This causes the air to be in an oscillatory motion, comparable to the oscillatory motion of water waves. There are, however, a few important differences. Firstly, water waves need gravity as the restoring force to sustain the oscillation. Sound does not need that. The restoring force is internally generated. Secondly, the direction of propagation of the water waves (parallel to the free surface) is perpendicular to the direction of the oscillation (perpendicular to the water surface). Sound waves do not behave so: the direction of propagation is parallel to the direction of the oscillations. The water waves are part of the family of transversal waves, whereas the sound waves belong to the longitudinal waves. The difference is illustrated in Figure 3.1).

longitudinal & transversal waves

The sound is characterized by the wave length, λ, and the speed of propagation, c. One could also use the frequency f which is related to the other two via $\lambda \cdot f = c$. The speed of propagations of the 'waves' is usually called the speed of sound. For air at room temperature this is some 340m/s. Sound in water propagates at a much faster speed, around 1500m/s. In metals this speed is even much greater.

speed of sound

Intermezzo: Harmonic oscillators

The basics of waves can be best understood from a mass-spring system. Consider a suspended spring from the ceiling with a mass, m, at the free end.

We will ignore gravity, as it only complicates the formulation, and does not do anything to the oscillation. The spring is characterized by its spring constant, k, and its equilibrium length, l_0. If the string is either extended by an amount x, or compressed by the same amount, it will exert a force that tries to restore the equilibrium length. The force is proportional to x and is given by:

$$F_s = -k \cdot x \tag{3.1}$$

The minus sign in the above equation shows that the spring's force is opposing the action that extended or compressed it.

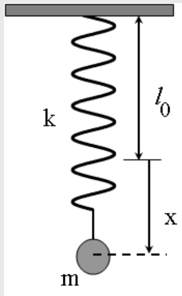


Figure 3.2 Mass suspended on a spring.

Suppose we have displaced the mass by an amount $x > 0$ and release it. The spring exerts a force on the mass that will start moving. The equation of motion is given by Newton's second law:

$$m\frac{d^2x}{dt^2} = -k \cdot x \rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{3.2}$$

The solution of this equation is:

$$x(t) = A\sin\omega_0 t + B\cos\omega_0 t \;\text{ with } \omega_0^2 = \frac{k}{m} \tag{3.3}$$

The constants A and B follow from the initial conditions of the problem. Indeed, we find that the mass starts a sinusoidal oscillation with a fixed frequency that depends only on the ratio of the spring constant and the mass: the weaker the spring or the heavier the mass, the slower the oscillation.

In air, the speed of sound is a function of the temperature only:

$$c = \sqrt{\gamma\frac{RT}{M}} \tag{3.4}$$

with $\gamma = 1.4$, $R = 8.3144$J/molK, $M = 28.8 \cdot 10^{-3}$kg/mol the mean molar mass of air and T the temperature. Note that the speed of sound of does not depend on the pressure. If we substitute $T = 20$°C, we find c = 344 m/s. The derivation of this result is rather complicated, it is given below.

3.1.1 Speed of sound in air

As mentioned above, sound waves in air are longitudinal waves, i.e. the wavy motion around the equilibrium position is parallel to the direction of the wave propagation and no longer perpendicular. The analysis we will follow is given in [1]. Consider a gas of given constant temperature and pressure. Then also the density is constant. We focus on a small portion of the gas between $\{x, x+\Delta x\}$. Due to a small perturbation all gas particles move a distance $\chi(x,t)$ along the x-direction. This function χ is the displacement from the equilibrium. Thus the gas originally at x will move to $x+\chi(x,t)$ and that at $x+\Delta x$ to $x+\Delta x+\chi(x+\Delta x,t)$. This is illustrated in Figure 3.3).

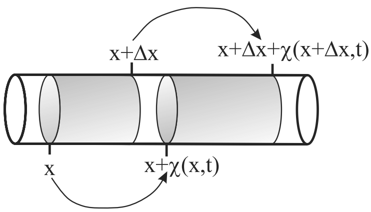


Figure 3.3 Gas between $x+\Delta x$ gets displaced due to a small perturbation.

Due to the motion of the gas, the density is no longer the same. Originally the mass was between x and $x+\Delta x$, now it is between $x+\chi(x,t)$ and $x+\Delta x+\chi(x+\Delta x,t)$. But since mass is a conserved quantity, the amount stays the same. With the help of ρ_0, the original, unperturbed density of the gas, we can write the conservation of mass as:

$$\rho_0 \Delta x = \left(\rho_0 + \Delta\rho\right)\left[\Delta x + \chi(x+\Delta x,t) - \chi(x,t)\right] \tag{3.5}$$

If we simplify this equation, using $\chi(x+\Delta x,t) - \chi(x,t) \approx \frac{\partial\chi}{\partial x}\Delta x$ and neglect higher order terms, i.e. products in Δx and $\Delta\rho$), we find that the change in density is given by:

$$\Delta\rho = -\rho_0 \frac{\partial\chi}{\partial x} \tag{3.6}$$

This is step 1.

The next step is relating this density change to a change in pressure. This is simple as the pressure is a function of the density and vice versa. Of course, the pressure is also a function of the temperature, as we know from e.g. the ideal gas law. So, $p = f(\rho,\cdots)$ and we can formally write: $\Delta p = \left(\frac{\partial p}{\partial\rho}\right)\Delta\rho$, without specifying what the other variables are that we have to keep constant when taking the derivative. This is step 2.

The last step is setting up Newton's law for the motion of the gas that was between x and $x+\Delta x$. This gas moves because a pressure from the left side is pushing it to the right, whereas from the right side the pressure is trying to push it back. Sound is a manifestation of the unbalance of these two pressure forces. The situation is depicted in Figure 3.4.

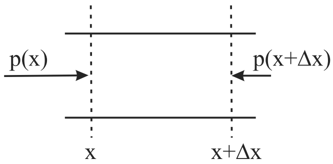


Figure 3.4 Unbalance in pressure on left and right side leads to motion.

Here, we call the area perpendicular to the x-axis A. Then the total mass of the gas in the interval $\{x, x+\Delta x\}$ is $\rho_0 \cdot A\Delta x$. The pressure on the left exerts a force $+Ap(x,t)$ and that on the right $-Ap(x+\Delta x,t)$. Thus, when we use that the acceleration a is the second derivative of the displacement χ: $a = \frac{\partial^2 \chi}{\partial t^2}$, Newton's equation of motion reads as:

$$\begin{aligned} \rho_0 \cdot A\Delta x \frac{\partial^2 \chi}{\partial t^2} &= Ap(x,t) - Ap(x+\Delta x,t) \\ &= -A\frac{\partial p}{\partial x}\Delta x \end{aligned} \tag{3.7}$$

If we simplify this equation we may write:

$$\begin{aligned} \rho_0 \frac{\partial^2 \chi}{\partial t^2} &= -\frac{\partial p}{\partial x} \\ &= -\frac{\partial p}{\partial \rho} \cdot \frac{\partial \rho}{\partial x} \\ &= \rho_0 \frac{\partial p}{\partial \rho} \cdot \frac{\partial^2 \chi}{\partial x^2} \end{aligned} \tag{3.8}$$

Where, for the last equality we have used the outcome of step 1. So, we arrive at

$$\frac{\partial^2 \chi}{\partial t^2} = \frac{\partial p}{\partial \rho} \cdot \frac{\partial^2 \chi}{\partial x^2} \tag{3.9}$$

wave equation

This is the wave-equation, which has similarities with the harmonic oscillator. The role of the mass is played here by the air that is in motion, the restoring action of the spring is provided by the pressure. As we can see, the two are found in a $\frac{k}{m}$ type of relation, but now as the change of the pressure with changing density. This is the compressibility. We could also say that air mass will try to continue its motion: it has inertia. But when an air parcel keeps moving, it will compress what is in front of it and that will cause an increase in the pressure that will resist a further compression and will try to stop the moving parcel. The same mechanism is active in fluids like water, when sound is generated.

The general solution of eq.(3.9) is: the density variations are like $\sin(\omega t - kx)$ which can be checked by substitution:

$$\chi(x,t) = \sin(\omega t - kx) \rightarrow$$
$$\frac{\partial^2 \chi}{\partial t^2} = -\omega^2 \sin(\omega t - kx) \quad (3.10)$$
$$\frac{\partial^2 \chi}{\partial x^2} = -k^2 \sin(\omega t - kx) \quad (3.11)$$

Putting this into eq.(3.9) gives:

$$c_s^2 \equiv \left(\frac{\omega}{k}\right)^2 = \frac{\partial p}{\partial \rho} \quad (3.12)$$

In this equation we have already used that the speed of sound is defined as $\frac{\omega}{k}$. The angular frequency, ω, is related to the frequency itself by means of $\omega = 2\pi f$. The quantity k is the wave number and is related to the inverse of the wavelength: $k = \frac{2\pi}{\lambda}$. Thus $\frac{\omega}{k} = f \cdot \lambda = c$.

Speed of sound in air
Now we can calculate the speed of sound in air. If we assume that air is an ideal gas and that sound is an isothermal phenomenon, we have:

$$p = \frac{RT}{M}\rho \rightarrow \left(\frac{\partial p}{\partial \rho}\right)_T = \frac{RT}{M} \quad (3.13)$$

with M the molar mass of air.
Thus, we would find that $c_s = \sqrt{\frac{RT}{M}}$. Conclusion (1), the speed of sound in air is only depending on the absolute temperature! Conclusion (2) its value at 300K is 294m/s, which is obviously much too low! The reason is that sound is not an isothermal process, but adiabatic. The expansion and compression (i.e. changes in density!) are too fast for thermal processes. Thus, like we have done before, the relation $pV^\gamma =$ const should be used, or written as a relation between the pressure, p, and the density, ρ, of the gas: $p =$ const $\cdot \rho^\gamma$. If we use this relation in eq.(3.12) we obtain:

$$c_s = \sqrt{\gamma \frac{RT}{M}} \quad (3.14)$$

For air $\gamma = 1.4$, thus at $T = 20°\mathrm{C}$ we calculate: c_s = 344m/s.

3.1.2 Donald Duck like voice

After the tragedy of the Russian submarine 'Koersk', the news on tv showed Norwegian and British divers who had to stay in a pressurized cabin for several days in order to avoid dangerous decompression effects. When spoken to, the divers answered with a Donald Duck like voice. Most likely, their cabin was not only pressurized but the gas inside is no longer standard air, as the speed of sound and therefore the sound of the divers' voices does not depend on the absolute value of the pressure!

This effect can easily be demonstrated if helium gas is available, e.g. from a helium filled balloon. After inhaling helium gas ones voice start to sound like that of Donald Duck. Why? Well, first of all the speed of sound in helium gas is different from that in air. According to the equation above, we have for He-gas ($\gamma = 5/3$, $M = 4 \cdot 10^{-3}$kg/mol) at 20°C: c_s = 1007 m/s! Much higher than in air due to the rather low molar mass M. But why does this influence the sound we make? This is because we form sounds by using our mouth as a kind of resonance-box. This means that we fix the wavelength, λ, of the sound produced. The frequency we hear follows from $f = c_s/\lambda$. Consequently, when we inhale helium, the frequency is about three times as high if we speak in the same way. Furthermore, the amplitude of the overtones is different for the different gases. Therefore, the sound is Donald Duck like.

3.2 Musical Instruments

Musical instruments generate sound and if played the right way this gives us pleasure. We can divide them into various categories:

- Stringed instruments, like the violin;
- Percussion, like the drum;
- Woodwinds, like the flute & brass instruments, like the trompet.

3.2.1 Stringed instruments

Let's start with the violin. In essence, it is a set of suspended strings, fixed at both ends.
The violin player will set a string in motion in a direction perpendicular to the string. This causes a higher tension in the string, as it is now longer than it was. Moreover, a net force is generated that will try to push the deformed string back into the original straight position, see Figure 3.6. Once the string is released, this force will set it in motion and the string will eventually reach its equilibrium position, where the net force is zero again. But just like in the case of the mass on a spring, the string has inertia (as it has a finite mass) and according to Newton's laws will continue to move, thereby stretching itself. Thus a force in the opposite direction will develop, slowing down the string until it comes to rest. However, the force is still non-zero and pointing towards the equilibrium position. Thus the string will be accelerated back and the oscillation will continue. The oscillation of the string, of course, also sets the air into an oscillating motion. The air will carry the sound wave, with exactly the frequency of the string and we will hear a tone.

Intermezzo: Sound in Newtonian fluids

Fluids like water and air belong to the class of Newtonian fluids. Newtonian fluids show a simple type of internal friction that is directly proportional to velocity differences within the fluid. Moreover, the proportionality constant, which is the dynamic viscosity μ, is a constant, it does not depend on the velocity of the fluid. For Newtonian fluids, the governing equations are the Navier-Stokes equations:

$$\begin{aligned} \text{Mass Balance} \qquad & \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} &=& \; 0 \\ \text{Momentum Balance} \qquad & \frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot \rho \vec{v}\vec{v} &=& \; -\nabla p + \mu \Delta \vec{v} \end{aligned} \tag{3.15}$$

We consider small perturbations of the state of rest specified by $\rho = \rho_0, p = p_0, \vec{v} = 0$. The small perturbations cab be written as:

$$\delta\rho = \rho - \rho_0 \qquad \delta p = p - p_0 \qquad \delta\vec{v} = \vec{v} - 0 = \vec{v} \tag{3.16}$$

The mass balance in linearized form gives:

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta\vec{v} = 0 \tag{3.17}$$

And for the linearized momentum balance we get

$$\rho_0 \frac{\partial \delta\vec{v}}{\partial t} = -\nabla \delta p + \mu \nabla^2 \delta\vec{v} \tag{3.18}$$

Obviously, we need an extra equation to eliminate p. For this the appropriate equation of state can be used. We will assume here that the wave motion generated by the perturbations is adiabatic (rather than isothermal). So, we can formally write for the equation of state (with s denoting the entropy):

$$p = p(\rho, s) \tag{3.19}$$

And thus for adiabatic conditions we can establish a relation between the small perturbations of pressure and density:

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c_f^2 \delta\rho \tag{3.20}$$

We can now eliminate δp and $\delta\rho$ from the equations of motion. Besides, we will drop the δ in front of the variables. This gives

$$\frac{\partial^2 \vec{v}}{\partial t^2} = c_f^2 \nabla^2 \vec{v} + \frac{\mu}{\rho_0} \nabla^2 \frac{\partial \vec{v}}{\partial t} \tag{3.21}$$

If we neglect the dissipation, we have the well-known wave equation $\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0$, which describes propagating waves with velocity c. Thus, we find as expected that waves of small amplitude in fluids propagate at the 'speed of sound' $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

Intermezzo: transversal waves in fluids?

Transversal waves are waves that have their amplitude perpendicular to the direction of propagation. Thus the velocity field in a fluid for a transversal wave moving in the x-direction which oscillates in the y-direction can be written as:

$$v_x(x,t) = 0 \qquad v_y(x,t) = \hat{v}_y \sin(\omega t - kx) \qquad v_z(x,t) = 0 \tag{3.22}$$

with v_y the component of the velocity in the y-direction; $\hat{v}_y$ is the (constant) amplitude. Thus we see, that for this transversal oscillation the velocity is only depending on (x,t) and not on y. This oscillation is similar to the transversal wave of Figure 3.1.
As this is a sound wave in a fluid, also the pressure will oscillate, with a similar dependence on (x,t):

$$p(x,t) = \hat{p} \sin(\omega t - kx) \tag{3.23}$$

However, the amplitudes of the sound wave, e.g. $\hat{v}_y$ and $\hat{p}$ are not independent from each other. The wave should still obey the momentum equation (3.15). We will ignore the frictional part:

$$\rho_0 \frac{\partial v_y}{\partial t} + \underbrace{\frac{\partial p}{\partial y}}_{=0} = 0 \rightarrow \hat{v}_y = 0 \tag{3.24}$$

So, we find that the amplitude of the transversal wave is zero: a Newtonian fluid can not sustain traveling transversal waves. This seems reasonable: the oscillating motion in the y-direction of the transversal wave needs a fluctuating pressure in the y-direction, which is not present in a transversal wave.
If we would take the viscous friction into account, we would find that a transversal wave is dampened so quickly that it can not penetrate into the fluid.

The frequency is determined by the wavelength, λ, of the oscillation and the speed of the oscillation. The latter is a function of the tension, F, in the string and the mass of the string per unit length, μ: $c^2 = \frac{F}{\mu}$.
As the string is fixed at its end points, the wavelength is also fixed. The end points of the wave are always at the equilibrium position. Actually, the string will always vibrate as a standing wave, *i.e.* the oscillation does not travel away from where it started. The standing waves have to fit in between the fixed end points. Thus, the wavelength can only be such, that the fixed ends of the string are fixed points of the standing wave. This is illustrated in Figure 3.7.

standing wave

Figure 3.5 A violin: suspended strings.

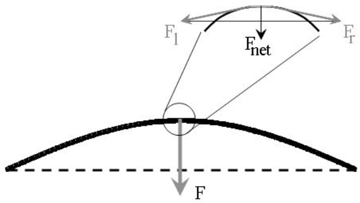


Figure 3.6 Net force on a string.

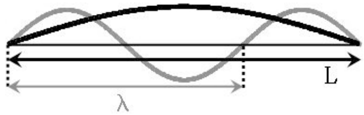


Figure 3.7 Allowed wavelengths.

The figure shows that the maximum wavelength is twice the length, L, between the fixed points, or rephrased $L = \frac{1}{2}\lambda_{max}$. Any multiple of $\frac{1}{2}\lambda$ that fits is also allowed. Thus, we see that the violin selects wavelengths according to:

$$L = n \cdot \frac{1}{2}\lambda \rightarrow \lambda = \frac{2L}{n} \tag{3.25}$$

If we combine this with the relations $\lambda \cdot f = c$ and $c = \sqrt{\frac{F}{\mu}}$, we find for the frequency of a violin string:

$$f = \frac{n\sqrt{\frac{F}{\mu}}}{2L} \tag{3.26}$$

From this we see that:

- the longer the string, the lower the tone;
- the heavier the string (which usually implies a thicker one), the lower the tone;
- the higher the tension, the higher the tone.

3.2.2 Ground frequency and overtones

In the above, the frequency is not fixed but goes in multiples of a lowest frequency $f_0 = \sqrt{\frac{F}{\mu}}/2L$, which is called the ground-frequency (found at $n = 1$ in eq.(3.26)). If the string is brought into vibration, also the other frequencies may be excited (*i.e.* with $n = 1, 2, 3, \cdots$). These tones are called the overtones. The ratio of the amplitudes of the ground-frequency and the overtones decides the 'color' of the tone. Each instrument does that in a different way. Therefore, the A on a flute sounds different from that of a trumpet, etc. The amplitudes of the excited frequencies are partly determined by the instrument (and partly of course by the skills of the musician). For instance, the violin is more than just a set of strings. The sound-box is equally important. It amplifies the vibrations of the strings, but does that according to its own, rather complicated rules. The sound box is also brought into resonance, with its own particular features. The difference between a high class violin and a standard one is found here; the former is made such that the tones sound better.

ground frequency
overtones

Obviously, the same holds for other instruments like the guitar. In Figure 3.8 the resonance pattern of the guitar is made visible for two different notes. The lines show the standing wave pattern of the 'box' of the guitar.

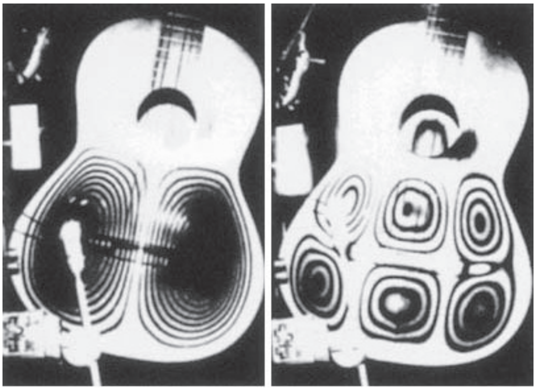


Figure 3.8 Standing waves on the box of a guitar.

3.2.3 Woodwinds

Although the organ is notan example of the woodwinds, its pipes create sound much in the same way as the flute does. In essence, the pipe is a hollow tube of a given length, that is open at one end and closed at the other.

turbulence

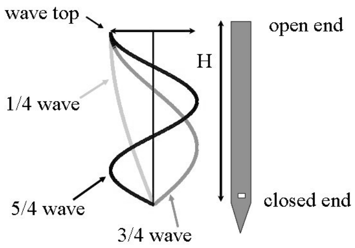
Air is blown in at the close end. A small slit close to the closed end lets the air escape. This slit has sharp edges over which the outgoing air generates turbulence. This turbulence can be seen as a collection of vibrations with a broad range of frequencies. These vibrations form the trigger that sets the air column in the organ pipe into motion. Again the organ pipe will 'select' the right frequencies at which resonance can occur. The idea is now, that at the open end the pressure is equal to that of the atmosphere and can not oscillate there. In the pipe, of course, the air will oscillate. This means that at the open end the standing wave must be such that for the pressure it acts as if the pressure is at a fixed

Figure 3.9 The large organ of Passau, Germany.

end, like the violin string. If the pressure is fixed, then the air velocity will be at a point of maximum amplitude. At the other end, the organ pipe is closed and the velocity will feel it is 'fixed'.

In Figure 3.10 some possible standing waves (from the point of view of the air velocity) are drawn. We see that the largest wavelength is 4 times the height of the pipe, H, *i.e.* $H = \frac{1}{4}\lambda_{max}$. The next wave can add half a wavelength, or two halfs, etc. Thus, the general rule for the wavelength in the organ pipe is:

wavelength organ pipe



$$H = \left(\frac{1}{2}n + \frac{1}{4}\right)\lambda \qquad (3.27)$$

Figure 3.10 Standing waves in an organ pipe.

The ground tone with $\lambda = 4H$ can produces very low tones. For instance a moderate size organ pipe with H = 2m will generate a wavelength of 8m. If we combine this with $\lambda \cdot f = c$ and recall that $c \approx 340$ m/s, we find that the ground frequency of this pipe is 42.5 Hz!

Music from a bottle

An empty or partly empty bottle can be turned into a musical instrument. By blowing over the opening some turbulence is created and standing waves are triggered in the bottle.

Again the standing waves are formed according to the same rules: the bottom of the bottle enforces the air velocity to be zero there; the opening causes the amplitude of the pressure fluctuations there to be zero. Consequently, we have the same situation as with the organ pipe. The tone of the 'whistling bottle' can easily be adapted: just fill the bottle partly with water and the height of the air column will decrease. This will make the frequency of the tone go up.

Figure 3.11 Sound from a bottle.

Sound of a tapped bottle

If we fill a bottle (or a glass) with water and we tap it, it will also produce sound. Now, however, the tone gets lower, rather than higher, when we fill the bottle more. The reason is, that by the tapping, we now have triggered the oscillation in the liquid instead of in the air. So, now the standing wave is in the liquid. Thus, the higher the bottle is filled, the longer the wavelength becomes and the lower the tone we hear.

3.2.4 Percussion

The drum, as an example of a percussion instrument, is a hollow cylinder with a thin, elastic sheet stretched over it.

Figure 3.12 Kettle drums

When a drum stick hits the drum sheet, the sheet is brought out of equilibrium. As with a string, a restoring force will evolve. Again, like the string, the sheet will move back, pass due to its inertia the equilibrium position and the restoring force will build up again. Similar to the violin, the sheet will find its resonance frequencies and respond accordingly. The cylinder below acts as an amplifier. Note that now the problem is 2-dimensional, whereas the string was only 1-dimensional. The eigen frequencies of the sheet are governed by the condition that at the edge of the sheet the amplitude of the oscillation is zero. Moreover, the tension in the sheet and the mass per unit area now determine the frequency, just like with the violin. So, for percussion to sound harmonic in e.g. an orchestra, also the percussionist will have to tune his or her instrument, by adjusting the tension of the sheet. If you come early to a performing orchestra, you will not only see and hear the violin players tune their instrument, but also the percussionists.

Similar to the violin string, the simplest oscillation of the sheet is sinusoidal in time. However, its spatial shape is more complicated: it is no longer a sinusoidal. Note, that the edge of the sheet is kept at a fixed position by the drum itself.

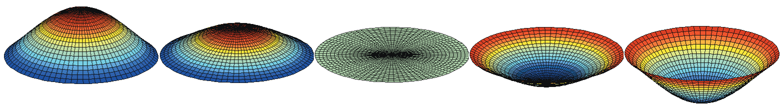


Figure 3.13 Oscillation sequence of the ground tone of a vibrating sheet (sequence runs from left to right and back, and so on).

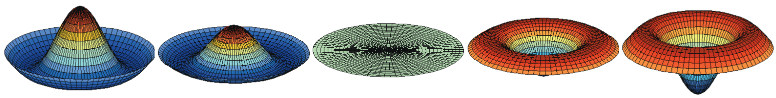


Figure 3.14 Oscillation sequence of the first over tone of a vibrating sheet (sequence runs from left to right and back, and so on).

The higher tones have a more complicated shape. For instance, the first over tone, given in Figure 3.14, shows more oscillations over the sheet. It looks like a Mexican hat. This is also quite comparable with the violin string: it is like one and a half sine.

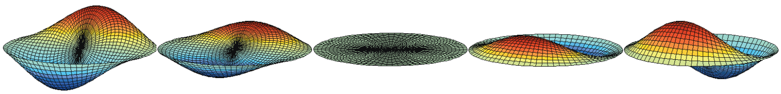


Figure 3.15 Oscillation sequence of the vibrating sheet with a diameter of the sheet as the symmetry axis.

We should, however, realize that in contrast to the string that can vibrate only in one dimension, the sheet has much more freedom. It could, for instance, also oscillate around a line through its center. The first tone that does so is given in Figure 3.15).

The frequency of the tones is found from a relation like that of the violin string. However, the overtones are not multiples of the ground frequencies. For the three oscillation modes we discussed above, the frequencies are: $f_0 = 0.3827\frac{\sqrt{T_s/\rho_S}}{R}$ with T_s the tension in the sheet, ρ_s the mass of the sheet per unit area and R the sheet diameter; $f_1 = 0.8785\frac{\sqrt{T_s/\rho_S}}{R}$; $f_2 = 0.6098\frac{\sqrt{T_s/\rho_S}}{R}$. As with the violin string: the higher the mass, the lower the tone ('heavy is slow'), the tenser the sheet the higher its pitch, the bigger the sheet (and thus the drum) the slower the oscillation and thus the lower the tone. Finally, overtones have a higher frequency.

The ear

If the drum is played, it will automatically lock on those oscillations that form the standing waves on the sheet. In other words, no matter what you do: the sound it makes is governed by the ground tone and its overtones, which is a discrete subset of all possible frequencies. This holds for all instruments. It is interesting to think for a moment on the reverse. Suppose we would send oscillating air with a wide range of frequencies to the drum sheet or the violin string. What would happen? Well, the sheet and string will 'pick out' its ground tone and overtones and start resonating with these frequencies (they are called the eigenfrequencies, for obvious reasons). *eigenfrequency*

Our ear also has a sheet or membrane that transfers sound (i.e. vibrating air) into signals for our brains. Why are we not affected by this resonance? Why can we hear frequencies over a wide range and do we not tune into the discrete set of eigen frequencies. The answer is in the sheet of our ear. It does not respond in a linear way to the vibrations of the air. It is this linearity that causes the drum sheet and the violin string to produce the sharp frequencies and only those (if played correctly). Luckily our ear is different. But on the other hand, the non-linearity has enormous consequences for physicists: most non-linear systems can not be solved with analytical methods, we have to use computer simulations and approximation techniques to study these systems.

3.3 The physics of an oscillating string

We return to the oscillating string and analyze in more detail its motion. To this end, we will treat the string as consisting of N beads, each with a mass m, that are coupled to one another by $N+1$ identical, massless pieces of string, see Figure 3.16.

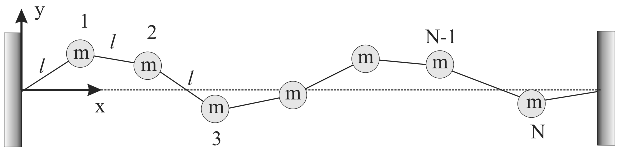


Figure 3.16 Discrete representation of an oscillating string.

Each bead feels the force of two string-parts left and right. We need to apply Newton's law for each of the beads, so let's concentrate on bead i. Each bead is displaced vertically by a distance Δy_i from its equilibrium position. The initial tension in the string is F. Due to the displacement, the strength might have changed a bit and the direction of the force on bead i is no longer parallel to the x-axis. The y-component of the force on bead i by string $i-1$ (see Figure 3.17) is given by:

$$\begin{aligned} F_{y,i-1} &= -(F+\Delta F)\sin\alpha_{i-1} \\ &= -\left(F+\frac{\partial F}{\partial l}\Delta l_{i-1}\right)\frac{y_i-y_{i-1}}{l+\Delta l_{i-1}} \\ &= -\frac{F}{l}(y_i-y_{i-1})+h.o.t. \end{aligned} \tag{3.28}$$

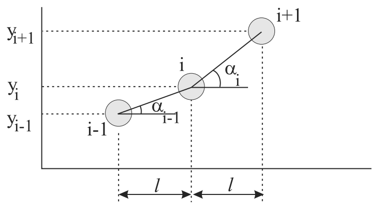


Figure 3.17 Three consecutive beads on the string.

Similarly, for the y-component of the force by string i we find:

$$F_{y,i} = \frac{F}{l}(y_{i+1}-y_i)+h.o.t. \tag{3.29}$$

Thus for the motion of bead i in the y-direction we find:

$$m\frac{d^2y_i}{dt^2} = \frac{F}{l}\left(y_{i-1}-2y_i+y_{i+1}\right) \tag{3.30}$$

We have this equation for all beads i, with the restriction that $y_0 = 0$ and $y_{N+1} = 0$ as these are the fixed points of the bead string. The set of N equations for the N beads can be solved by realizing that we are looking for small-amplitude oscillations around an equilibrium, thus the form $y_i(t) = A_i \sin \omega t$ can be introduced. A complete analysis can be found in e.g. [3].

For us, it is more important to see what happens if the number of beads increases and the length of each piece of string goes down: we turn the bead-string analysis into a continuous description that deals with the original string and its oscillations around an equilibrium position. More precisely, we take $N \to \infty$ and thus $l \to 0$ such that:

$$\begin{aligned} \lim_{N\to\infty} \sum_{i=1}^{N} m_i &= M \\ \lim_{N\to\infty} \sum_{i=1}^{N} l_i &= L \end{aligned} \tag{3.31}$$

As a result, the set $\{y_i\}$ changes to a continuous function, $y(t)$. The differences that appear on the right hand side of eq.(3.30) will change to derivatives with respect to the horizontal coordinate. This can be seen as follows. Consider the position of the consecutive beads, like in Figure 3.18).

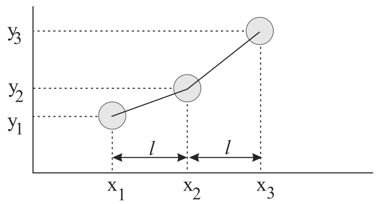


Figure 3.18 Three consecutive beads on the string.

First we consider $y_2 - y_1$ in the limit $l \to 0$ and replace the discrete variables y_1, y_2 etc. by the continuous variables $y(x_1), y(x_2)$ etc.:

$$\begin{aligned} \frac{y_2 - y_1}{l} &= \frac{y(x_2) - y(x_1)}{l} \\ &= \frac{y(x_{12} + l/2) - y(x_{12} - l/2)}{l} \\ &\to \frac{\partial y}{\partial x}(x_{12}) \quad \text{for} \quad \lim_{l\to 0} \end{aligned} \tag{3.32}$$

where $x_{12} = \frac{x_1 + x_2}{2}$. Similarly we have:

$$\lim_{l\to 0} \frac{y_3 - y_2}{l} = \frac{\partial y}{\partial x}(x_{32}) \tag{3.33}$$

Finally, combining the two equations (3.32) and (3,33) we obtain:

$$\lim_{l\to 0}\frac{\frac{\partial y}{\partial x}(x_{32})-\frac{\partial y}{\partial x}(x_{12})}{l} = \lim_{l\to 0}\frac{\frac{\partial y}{\partial x}(x_2+l/2)-\frac{\partial y}{\partial x}(x_2-l/2)}{l} \equiv \frac{\partial^2 y}{\partial x^2}(x_2) \tag{3.34}$$

Summing up all these 'mathematical tricks' and defining $\mu \equiv \frac{m}{l}$ the equation for the y-component of the string as a function of $\{x,t\}$ is:

$$\frac{\partial^2 y}{\partial t^2}=\frac{F}{\mu}\frac{\partial^2 y}{\partial x^2} \tag{3.35}$$

wave equation

This is the well-known wave equation, that describes oscillations or waves of a continuous medium, like our string. Solutions of the form $y(x,t)=A\sin(\omega t-kx)$ obey this equation, as is seen by inserting this into the wave equation. Notice that the angular frequency ω and k are related by way of $\left(\frac{\omega}{k}\right)^2=\frac{F}{\mu}$. The coefficient k is called the wave number and is a function of the wavelength λ: $k=\frac{2\pi}{\lambda}$. From the solution we see, that at every point x the string oscillates with the same frequency (and the same amplitude) but that the phase of the oscillation is different from point to point. Or alternatively, at every time t the shape of the string is sinusoidal. These waves 'move' along the string and their speed is given by $c=\frac{\omega}{k}$, which is obvious when one writes the solutions as $y(x,t)=A\sin\left(k\left[x-\frac{\omega}{k}t\right]\right)=A\sin(k[x-ct])$. Notice that thus the speed c is given by $c^2=\frac{F}{\mu}$, an equation that has two solutions ($c=\pm\sqrt{F/\mu}$) showing that the waves can run to the left or to the right.

So, we found the general solution of the oscillations in the string. If we compare this to the standing wave of the violin string, we see that the oscillations induced on the string run up and down the string and reflect at the fixed ends. Then, they will interfere with the oscillations that are still traveling towards the end. Only those that exactly fit according to the standing wave rules we discussed earlier will survive this interference and get amplified, all others are quickly dampened to such amplitudes that we can not hear.

3.4 The sound of mountain streams

Every small stream that is flowing not too smoothly but also not too wildly, seems to make the same sound, a characteristic kind of babbling or whispering. Where does this come from? Why don't large rivers in low lands make the same 'noise'?

sound from bubbles

The sound is caused by small bubbles that are captured in the streams for a while before they escape again. These bubbles are air trapped by fluctuations of the water surface or tiny water falls. In order to understand this we have to think about bubbles as a source of sound.
Why or how do bubbles in water generate sound? If the pressure inside the bubble

is in equilibrium with its surrounding and no perturbation is present, no sound will be generated.
However, the bubble-water system can be seen as a mass-spring system. If the spring is not at its equilibrium length, it will exert a force on the mass and cause the mass to move. Obviously, due to inertia the mass will continue to move even if the spring is at its equilibrium position and thus an oscillation sets in. The spring is represented by the pressure of the gas. The inertia is the water surrounding the bubble. Then what is the velocity? It is not the center of gravity of the bubble that is oscillating, it is the size of the bubble that does so. Minnaert studied this effect in 1933 and tried experimentally to find the frequency of the oscillating bubbles by comparing it with the sound of a tuning fork (see [9]).

Figure 3.19 A small river in the mountains.

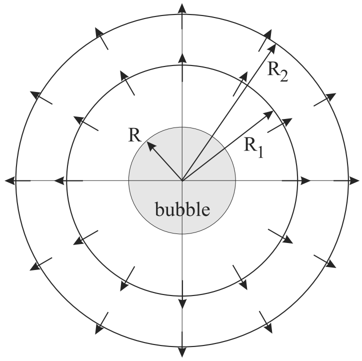


Figure 3.20 The amount of water between $R_1 < r < R_2$ is constant as water is incompressible.

What we need to do, is to derive an equation for the oscillating bubble that looks like a mass-spring system. this will give us the oscillation frequency as well. Let's first think about the momentum in our bubble-water system. We put the center of the bubble in the origin of our coordinate system. Further, we assume that the bubble radius, R, oscillates around an equilibrium value R_0. Due to this oscillation both the gas inside the bubble and the water surrounding the bubble will oscillate. The momentum of the gas is negligible in comparison with that of the water as $\frac{\rho_{air}}{\rho_{water}} \approx 10^{-3}$. The momentum of the liquid can be calculated if we know the liquid velocity. This is, however, changing from place to place, since we can safely assume, that the liquid at infinity will not 'notice' the presence of a tiny bubble. The velocity *field* is obtained by realizing that the problem is spherical symmetric: in spherically coordinates, the velocity has only a radial component and no quantity can depend on the angular coordinates ϕ and θ.

Consider the water present between two spheres, one of radius R_1, the other of R_2 (see Figure 3.20).

Since water is incompressible, we have that the total amount, or better total *mass*, of water enclosed by the two spheres is constant. Thus, according to the balance equation, what flows in at $r = R_1$ must flow out at $r = R_2$. But we can use the symmetry. The velocity of the water has only a radial component, which may only depend on the radial coordinate: $v_r(r)$.
Into our volume $R_1 < r < R_2$ flows water with a velocity $v_r(R_1)$, the area through which it flows is $4\pi R_1^2$. Thus the inflow is $\rho_{water} 4\pi R_1^2 v_r(R_1)$. Similarly we find that at $r = R_2$ the outflow of water is: $\rho_{water} 4\pi R_2^2 v_r(R_2)$. Since the inflow must be equal to the outflow we have:

$$\rho_{water} \cdot 4\pi R_1^2 \cdot v_r(R_1) = \rho_{water} \cdot 4\pi R_2^2 \cdot v_r(R_2) = const \tag{3.36}$$

Since the size of R_1 and R_2 is arbitrary, we find that the total flow through any surface of a sphere with radius r must be the same constant. So, we find that $v_r(r) \propto \frac{1}{r^2}$. Furthermore, the motion of the water is a function of the oscillation of the bubble radius. We will call the velocity at the bubble surface ($r = R$) v_R. Thus we have for the velocity at any point in the water that it has only a radial component which is equal to:

$$v_r(r) = \frac{R^2}{r^2} v_R \tag{3.37}$$

Now we can calculate the total momentum of the liquid. Take the mass of water between two concentric spheres of radius r and $r + dr$, respectively. This mass is $dm = \rho_{water} \cdot 4\pi r^2 dr$ and it has a velocity of $v_r(r)$ in the radial direction. So we compute the momentum of the entire water mass as

$$\begin{aligned} P &= \int_R^\infty \rho_{water} \cdot 4\pi r^2 \cdot v_r(r) dr \\ &= \int_R^\infty \rho_{water} \cdot 4\pi r^2 \cdot \frac{R^2}{r^2} v_R dr \\ &= \rho_{water} \cdot 4\pi R^2 v_R \int_R^\infty dr \end{aligned} \tag{3.38}$$

But there is a problem: the integral of the last equation is divergent! What went wrong and how can we cure it? Well, we have been a bit sloppy with the momentum. It is actually a *vector* quantity, but in eq.(3.38) we have been summing the contribution of all water elements without paying attention to the direction of the momentum of each small piece of water mass. And as the velocity has only a radial component, so does the momentum. This means that the total momentum is zero! Of course, the same will hold for the force exerted by the pressure inside the bubble on the water. A way out of this is to make use of the total kinetic energy of the water and then find the momentum via $E_{kin} = \frac{1}{2} m v^2$. Thus,we calculate the kinetic energy (which is a scalar, so no problems expected):

$$\begin{aligned} E_{kin} &= \int_R^\infty 4\pi r^2 \cdot \frac{1}{2}\rho_{water} v_r^2(r) dr \\ &= \int_R^\infty 2\pi\rho_{water} v_R^2 \frac{R^4}{r^2} dr \\ &= 2\pi\rho_{water} v_R^2 R^3 \end{aligned} \tag{3.39}$$

So, using $P = mv = \frac{dE_{kin}}{dv}$ we find for the momentum of the water:

$$P = 4\pi\rho_{water} R^3 v_R \tag{3.40}$$

The force exerted by the bubble on the liquid is:

$$F = (p - p_0) \cdot 4\pi R^2 \tag{3.41}$$

in which p is the pressure in the bubble and p_0 the equilibrium pressure when the bubble has radius R_0.

Applying Newton's law thus gives:

$$4\pi\rho_{water} \frac{d}{dt} R^3 v_R = (p - p_0) 4\pi R^2 \tag{3.42}$$

What we need to do still, is to couple the velocity of the bubble-water interface, v_R, to the bubble radius. That is trivial as $v_R \equiv \frac{dR}{dt}$. Furthermore, we need to do the same for the pressure in the bubble. Obviously, the pressure is related to the bubble volume by the ideal gas law. However, we need to make an assumption for the temperature. Is the process isothermal or adiabatic or else? We assume that the oscillation is fast enough for an adiabatic process. Then, we can use (as we have done before) $pV^\gamma = p_0 V_0^\gamma$. This relates the pressure to the radius of the bubble. The final step is assuming that the amplitude of the oscillation is small, so we may neglect higher order terms. We write formally: $R(t) = R_0 + \Delta R(t)$, with $\Delta R / R_0 \ll 1$. Now we can express the pressure as a function of the radius:

$$\begin{aligned} \frac{p}{p_0} &= \left(\frac{V_0}{V}\right)^\gamma \\ &= \left(\frac{R_0^3}{(R_0 + \Delta R)^3}\right)^\gamma \\ &\approx 1 - 3\gamma \frac{\Delta R}{R_0} \end{aligned} \tag{3.43}$$

This means: $p - p_0 \approx -3\gamma p_0 \frac{\Delta R}{R_0}$, in other words the pressure difference is linear in the small radius-variation. So for evaluating the force at the right hand side of eq.(3.42), we need to take for the surface area $4\pi R_0^2$. The higher order terms may safely be neglected.

At the left hand side of eq.(3.42) we have to take the time derivative of $R^3 v_R$. This gives two contributions, $3R^2 v_R \frac{dR}{dt}$ and $R^3 \frac{dv_R}{dt}$. As we are looking for oscillations with small amplitude, we can safely assume that ΔR will be of the form:

$$\Delta R = A \sin \omega t \tag{3.44}$$

where the amplitude A is small with respect to R_0. So, the velocity v_R, which is equal to $\frac{dR}{dt}$, will be proportional to A and thus vary linearly with ΔR. Consequently, the term $3R^2 v_R \frac{dR}{dt}$ is quadratic in ΔR and can be neglected with respect to $R^3 \frac{dv_R}{dt}$ which is linear in ΔR: $R^3 \frac{dv_R}{dt} \approx R_0^3 \frac{d^2 \Delta R}{dt^2}$. If we put all this together, we find for eq.(3.42):

$$\frac{d^2 \Delta R}{dt^2} + \frac{3\gamma p_0}{\rho_{water} R_0^2} \Delta R = 0 \tag{3.45}$$

mass-spring

Thus, we have the type of equation similar to the mass-spring we were looking for. The solution of the variation of the bubble radius is:

$$\Delta R = A \sin \left(\sqrt{\frac{3\gamma p_0}{\rho_{water} R_0^2}} t \right) \tag{3.46}$$

The oscillation frequency is equal to

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{3\gamma p_0}{\rho_{water} R_0^2}} \tag{3.47}$$

For a bubble with a radius of 1mm, we find: $f = 3300Hz$, which is in the range of what we hear from the streams.

3.4.1 The sound of boiling water

Water in a whistling kettle, heated on a stove has its own characteristic sounds. Before it really boils the first sound can be heard. The kettle seems to click and hiss. Hardly any steam is seen to escape from the kettle. If we wait longer the hiss becomes louder. After a while, the water in the kettle is boiling and the sound made seems to be softer, less intense. What is happening here?

Well, the process of bringing water to the boil is perhaps more complicated than one might think. Of course, the actions taken to put water in the kettle and subsequently the kettle on the stove are simple enough. But what is happening to the water inside the kettle? The bottom of the kettle will transfer heat to the water, which, obviously, will get hotter. But the temperature of the outside of the bottom

Figure 3.21 Boiling water: bubbles breaking through the surface.

of the kettle will be pretty high, in any case much higher than the water temperature itself. After a while the mean temperature of the water will be close to 100°C. The inside of the kettle bottom is then above 100°C. So, locally the temperature of the liquid will also rise above 100°C. As this is the boiling temperature (at a pressure of 1 bar), the first small vapor bubbles will be formed. These are created at specific spots at the bottom plate, where a small roughness acts as a nucleus where bubbles can grow on. Similarly creation of bubbles from certain spots can be seen in a fresh glass of mineral water or beer. Here, usually a series of bubbles is formed. The creation of the bubbles is accompanied with the clicking sound.

3.4.2 Surface tension

The bubbles in the water are supersaturated, that is they are formed in a very thin layer of water that has a water temperature well above 100°C. Why are the bubbles first formed in that layer only? The reason for this super saturation is the higher pressure in a bubble in comparison with its surrounding due to the surface tension. The surface tension can be seen as a kind of film around a bubble or droplet that has 'elastic properties'. If one tries to stretch it, it will resist. More precisely, if one tries to increase the area of the film one has to supply extra energy to the system, proportional to the increase in the surface area. Moreover, if the surface is bend, it will exert a force that will try to straighten it. So, for a soap bubble, the soap film will act like the rubber of a balloon. The pressure inside the soap bubble is, like that in the balloon, somewhat higher than the pressure in the surrounding air. The extra pressure due to surface tension is given (for a spherical object) by Laplace's equation:

surface tension

$$\Delta p = \frac{2\sigma}{R} \tag{3.48}$$

with Δp the pressure difference between inside and outside of the object of radius R; σ is the coefficient of surface tension, or short: surface tension. It is a property of the two materials forming the surface.

Let's return to the boiling water. Here the effect of surface tension is much more important. The bubbles formed are very small, so the overpressure will be relatively large. If we take the initial size as 0.01mm, we can calculate $\Delta p = 1.4 \cdot 10^4$ Pa (taking $\sigma = 70$ mPas). From the steam tables we can find, that at a pressure of 1.15bar the saturation temperature of liquid water and water vapor is equal to $T_{sat} = 103°$ C. Thus, the bubbles formed are hotter than most of the water in the kettle. If they detach from the kettle bottom and rise upwards they will be surrounded by water that is colder. Thus, the bubbles will lose heat and collapse as they are now in an environment that is below the saturation temperature, even at the local pressure. This collapse is heard as the hiss. Of course, due to the collapse of the bubbles no bubbles reach the free liquid surface in the kettle and thus hardly any steam is produced.

Intermezzo: How thick is the film of a soap bubble?
Consider a soap bubble blown by a small child. For the soap bubble, the surface tension is in the order of $100 \cdot 10^{-3} Pa.s$. Thus, the overpressure in a soap bubble of 10cm is only 2Pa! Compared to the pressure of the surrounding air, this is negligible. The pressure inside and outside the soap bubble are thus virtually the same.
Why do most soap bubbles first rise and then come down again (if they 'live' long enough)? Obviously, the soap film around the bubble adds to the weight of the bubble. Nevertheless, the buoyancy force is initially greater than the force of gravity that tries to pull the bubble towards the ground. This is caused by the higher temperature of the air inside the soap bubble. This air is warmed by our longs. Let's use this to estimate the thickness of the soap film. We take a bubble with a diameter of 5cm. Further, we assume that the temperature of the air inside the bubble is half way between our body temperature and the surrounding air temperature. It is partly cooled down already and partly mixed with surrounding air when we blew it out of our mouth into the soap bubble. As we have seen, we can neglect safely any overpressure due to surface tension. Thus the buoyancy force is $\rho_{air}(T) g \frac{\pi}{6} D^3$, gravity's force is $-mg$. The mass of the bubble is $\rho_{air}(T+\Delta T)\frac{\pi}{6}D^3 + \rho_{soap}\pi D^2 \delta$, in which δ is the thickness of the soap film. Thus, for a floating bubble we find:

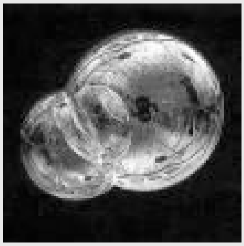


Figure 3.22 Soap bubbles.

$$\begin{aligned} \rho_{air}(T) g \tfrac{\pi}{6} D^3 - \left(\rho_{air}(T+\Delta T)\tfrac{\pi}{6}D^3 + \rho_{soap}\pi D^2 \delta\right) = 0 \rightarrow \\ \delta = \frac{1}{\rho_{soap}} \frac{\partial \rho}{\partial T} \Delta T \frac{D}{6} \end{aligned} \qquad (3.49)$$

In the above equation we have used that ΔT is much smaller than T.
For air we can use the ideal gas law to calculate $\frac{\partial \rho}{\partial T} = \frac{\rho}{T}$. Thus, for a soap bubble of $D = 5$cm, with $T = 300$K, $\Delta T = 10$ K, $\rho_{air} = 1.2$ kg/m^3 and $\rho_{soap} = 10^3$kg/m^3, we find: $\delta = 3 \cdot 10^{-7}$ m.

As the stove heats the water further, a point is passed at which the water temperature everywhere is at $100°C$ or above. Now boiling can be sustained everywhere and bubbles are formed continuously. Furthermore, they no longer collapse, but break through the free liquid surface. The sound becomes more continuous and softer. A splashing sound is heard that is due to the erupting bubbles throwing up some liquid water. Besides, steam is produced that leaves the kettle and might cause the kettle to whistle.

Boiling water in the microwave
This also provides an explanation of the explosive situation that can arise when heating a cup of water in a micro wave. Here the water is heated 'from the inside' due to the absorption of electro-magnetic waves by the water. So, the heating is rather uniform: no hot spots as compared to the heating in the kettle. Furthermore, the cup itself is heated indirectly and thus lags behind a bit in temperature. If the wall of the cup is smooth, no nucleation spots for the generation of small vapor bubbles may be present. Thus, the water can reach a temperature above the saturation temperature at the given pressure: the entire liquid is super saturated. This is rather dangerous as now the introduction of nuclei (e.g. by adding sugar, salt or any other powder) will set the entire liquid into a vivid boiling. Due to the formation of vapor the mixture level will swell with the danger of overflowing. Furthermore, the sudden expansion will 'blow' water out of the cup with the possibility of serious consequences.

3.5 The sound of a fresh, hot cup of instant coffee

A very peculiar sound effect can be found in a fresh cup of instant coffee. If the bottom of the cup is tapped by a spoon a clear sound is heard.
However, if the coffee is first stirred and then the bottom tapped, you will hear a lower pitch which is getting higher with time. After a while the pitch remains constant and of course the stirred coffee stops rotating. What is the reason of this changing pitch? Does warm water show the same? The answer to the last question is: no. So, changes in the distribution of the temperature in the liquid are apparently irrelevant. One could have thought that perhaps in the unstirred case the temperature of the liquid is not uniform, whereas in the stirred case it is. But the warm water experiment shows that this aspect does not account for what we hear.

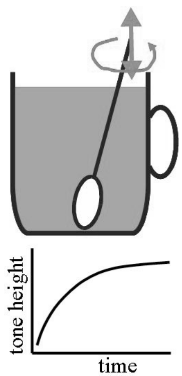


Figure 3.23 Tapping and stirring a cup of instant coffee.

Then, what is so special about the coffee? The answer is that small air bubbles have been trapped in the powder mass. They are so small that they can not escape the coffee easily. The bubbles, over time, will move close to the free surface of the coffee, forming a foam.
When stirred these bubbles will be dispersed again over the entire volume of the coffee. If the stirring stops, the velocity of the coffee will decrease and the bubbles will again move upwards to the surface. This suggests that the sound we hear stems from the presence of the bubbles in the liquid. And indeed, it does. To undestand what happens we have to consider waves in the coffee. The tapping will send

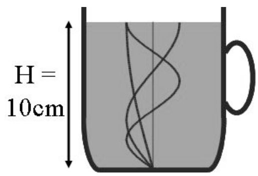


Figure 3.24 Possible standing waves in the cup.

waves into the liquid. As we have seen before, due

resonance

to resonance we will hear dominantly those waves that 'fit' into the liquid. If the height of the liquid is H, we can estimate the frequency from:

$$H = (n + \frac{1}{2}) \cdot \frac{\lambda}{2} \rightarrow \lambda = \frac{4H}{1 + 2n} \tag{3.50}$$

with λ the wavelength and $n = 0, 1, 2, \cdots$. Furthermore, the frequency and wavelength are coupled: $f \cdot \lambda = c_w$ (c_w is the speed of sound in water). Thus we have that the sound we hear has a frequency

$$f = \frac{c_w}{\lambda} = \frac{1 + 2k}{4} \frac{c_w}{H} \rightarrow f_0 = \frac{c_w}{4H} = 3750 Hz \tag{3.51}$$

where we have taken: $c_w = 1500$ m/s and $H \approx 10$ cm.

This reasoning also holds for the coffee with the bubbles. Here we will also hear the standing waves. Thus, we may conclude that the velocity of sound in a liquid with bubbles is different from that without! But why? To understand this we will have to look deeper into the phenomenon of sound and what determines the speed of sound in a given material.

Bubbles and the speed of sound

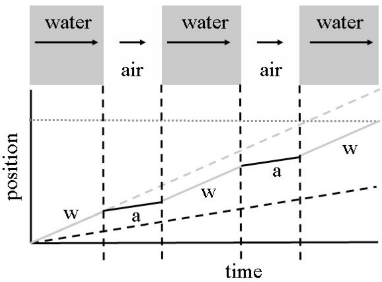


Figure 3.25 Average velocity of sound going through water and air alternatingly.

The sound we hear is caused by standing waves of which the frequency is determined by the height of the liquid level in the cup and the speed of sound in the coffee. The presence of small bubbles in the coffee has a major influence on the speed of sound in the coffee. We can now understand why. When we were dealing with the speed of sound in air we found that it is given if we know the relation between the pressure and the density. That still holds for water with bubbles. However, nature has a surprise. Most people would guess that the speed of sound in a mixture of air bubbles and water is somewhere

speed of sound in water and air

between the values of the speed of sound in air (340m/s) and in water (1500m/s). The reasoning would be something like: if the sound goes through the bubbles it will do so with a velocity of 340m/s. When it has passed through the bubble, it will continue its way through water at a speed of 1500m/s. Then it will propagate through a bubble and so on. The idea is visualized in Figure 3.25.

However, this reasoning is in many cases wrong. It does not take into account, that the wave is a 'coherent motion' of the system. So if the wavelength is large, i.e. much larger than the bubble size and the bubble-bubble distance, the sound

sees the bubbly mixture as a homogeneous 'liquid'. Think of the sound waves as the string-bead system. The strings were responsible for the restoring force, the beads had the inertia to keep the oscillation going. In this case water provides the inertia, but the bubbles can be compressed easily until finally their increasing pressure stops the water motion and tries to restore the equilibrium situation.
We have to find the relation between a change in pressure and resulting change in the density of the bubbly mixture. Our starting point is Figure 3.5). What we need to calculate is $c^2 = \frac{\partial p}{\partial \rho}$. But we may equally well calculate the inverse: $\frac{\partial \rho}{\partial p}$
We shall ignore the relative velocity between the gas bubbles and the liquid (or do the experiment in the space shuttle, were the influence of gravity is absent). The amount of gas present is quantified as the volume fraction of gas. That is the total volume of the gas bubbles divided by the total volume: $\alpha = \frac{V_{gas}}{V}$.

volume fraction of bubbles

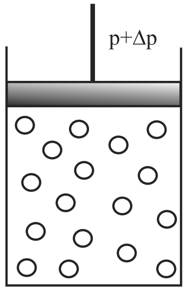


Figure 3.26 Compression of a bubble/water mixture.

If we increase the pressure by Δp, each of the phases will be compressed. However, they will not do so at the same rate: we may expect the gas phase to be much more compressed than the liquid phase. Thus we can expect that α will also change due to the compression. The changes in volume of each of the phases is given via the compressibility of each phase:

$$\begin{aligned} V_{gas} &= V_{gas,0} + \frac{\partial V_{gas}}{\partial p}\Delta p \\ V_{liq} &= V_{liq,0} + \frac{\partial V_{liq}}{\partial p}\Delta p \end{aligned} \tag{3.52}$$

and thus for the total volume after the compression:

$$V = V_{gas} + V_{liq} = V_0 + \left[\frac{\partial V_{gas}}{\partial p} + \frac{\partial V_{liq}}{\partial p}\right]\Delta p \tag{3.53}$$

in which subscript 0 indicates the situation before the compression. The change in the volume fraction of gas when the bubbly mixture is compressed, can then be calculated as follows:

$$\begin{aligned} \alpha &\equiv \frac{V_{gas}}{V} = \frac{V_{gas,0}\left(1 + \frac{1}{V_{gas,0}}\frac{\partial V_{gas}}{\partial p}\Delta p\right)}{V_0\left(1 + \frac{1}{V_0}\left[\frac{\partial V_{gas}}{\partial p} + \frac{\partial V_{liq}}{\partial p}\right]\Delta p\right)} \\ &\approx \frac{V_{gas,0}}{V_0}\left(1 + \frac{1}{V_{gas,0}}\frac{\partial V_{gas}}{\partial p}\Delta p\right)\left(1 - \frac{1}{V_0}\left[\frac{\partial V_{gas}}{\partial p} + \frac{\partial V_{liq}}{\partial p}\right]\Delta p\right) \\ &= \alpha_0\left(1 + (1-\alpha_0)\frac{1}{V_{gas,0}}\frac{\partial V_{gas}}{\partial p}\Delta p - (1-\alpha_0)\frac{1}{V_{liq,0}}\frac{\partial V_{liq}}{\partial p}\Delta p\right) \end{aligned} \tag{3.54}$$

where we have neglected higher order terms in Δp. From the above equation we obtain:

$$\frac{d\alpha}{dp} = \alpha(1-\alpha)\left(\frac{1}{\rho_{liq}}\frac{\partial \rho_{liq}}{\partial p} - \frac{1}{\rho_{gas}}\frac{\partial \rho_{gas}}{\partial p}\right) \tag{3.55}$$

In this equation we have dropped the subscript 0 and we have used $\frac{1}{V}\frac{\partial V}{\partial p} = -\frac{1}{\rho}\frac{\partial \rho}{\partial p}$. Now, we can almost calculate the speed of sound in a bubbly mixture. First we express the mixture density, ρ, in terms of the densities of the gas and liquid:

$$\begin{aligned} \rho &= \frac{M}{V} \\ &= \frac{M_{gas}+M_{liq}}{V} = \frac{V_{gas}\rho_{gas}+V_{liq}\rho_{liq}}{V} \\ &= \alpha\rho_{gas} + (1-\alpha)\rho_{liq} \end{aligned} \tag{3.56}$$

From this equation we calculate $\frac{\partial \rho}{\partial p}$:

$$\frac{\partial \rho}{\partial p} = (1-\alpha)\frac{\partial \rho_{liq}}{\partial p} + \alpha\frac{\partial \rho_{gas}}{\partial p} + \left(\rho_{gas} - \rho_{liq}\right)\frac{\partial \alpha}{\partial p} \tag{3.57}$$

speed of sound in bubbly flow

Finally, we insert the result we had for $\frac{\partial \alpha}{\partial p}$ into this equation, simplify and use the definition of the speed of sound: $c^2 = \frac{\partial p}{\partial \rho}$ or $\frac{1}{c^2} = \frac{\partial \rho}{\partial p}$. This gives us the speed of sound in our bubbly mixture:

$$\frac{1}{c^2} = \frac{(1-\alpha)\left[1-\alpha\left(1-\frac{\rho_{gas}}{\rho_{liq}}\right)\right]}{c_{liq}^2} + \frac{\alpha\left[1+(1-\alpha)\left(\frac{\rho_{liq}}{\rho_{gas}}-1\right)\right]}{c_{gas}^2} \tag{3.58}$$

In Figure 3.27 the mixture velocity is plotted as a function of the volume fraction of the gas bubbles. It is clear from the graph, that the speed of sound in a bubbly mixture drops dramatically below the speed of sound in air. We have to keep in mind that our result is only valid for 'long' wave, as we discussed above.
The volume fraction where the speed of sound of the mixture is equal to that of air is $\alpha^* \approx 0.11\%$. For larger volume fractions the velocity of sound is lower than in air. So, if we go back to the coffee problem, we can now understand that when bubbles are stirred into the coffee, the tone we hear from our tapping must be lower than in the clear case as $f \propto c$. Furthermore, after the stirring has stopped, the bubbles will gradually move to the surface of the coffee and thus the tone will rise gradually as well. This experiment is simple enough to do at home!

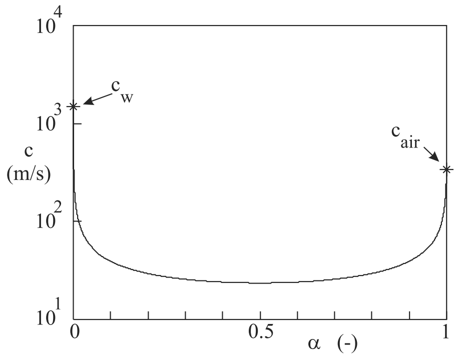


Figure 3.27 Speed of sound in an air-bubble/water mixture. Notice the log scale on the vertical axis. On the left vertical axis the speed of sound in water is given; on the right one that of air.