This file has been optimized for two-sided printing on standard A4 paper, page setup landscape and short edge binding.

Electromagnetic Waves – A Repetitive Guide
Electromagnetic waves appear in many forms and their applications are extremely widespread. Without exaggeration it may be said that our ability to employ and manipulate electromagnetic waves forms one of the reasons that communication plays such an important role in society.

The macroscopic theory of electromagnetic waves has been formulated by Maxwell in 1864. But the mathematical-physical nature of the subject makes it difficult for students to master even today. The continuous stream of new college textbooks shows that many teachers encounter this problem and attempt to resolve it by presenting the theory in some suitable form.

In the Electrical Engineering curriculum of the Delft University of Technology, the teaching of electromagnetic waves has been divided into three stages: 1) a basic course on Electricity and Magnetism, 2) an introductory course on Electromagnetic Waves, and 3) advanced courses on the application and computation of electromagnetic waves. The current booklet is written to facilitate the introductory course on Electromagnetic Waves.

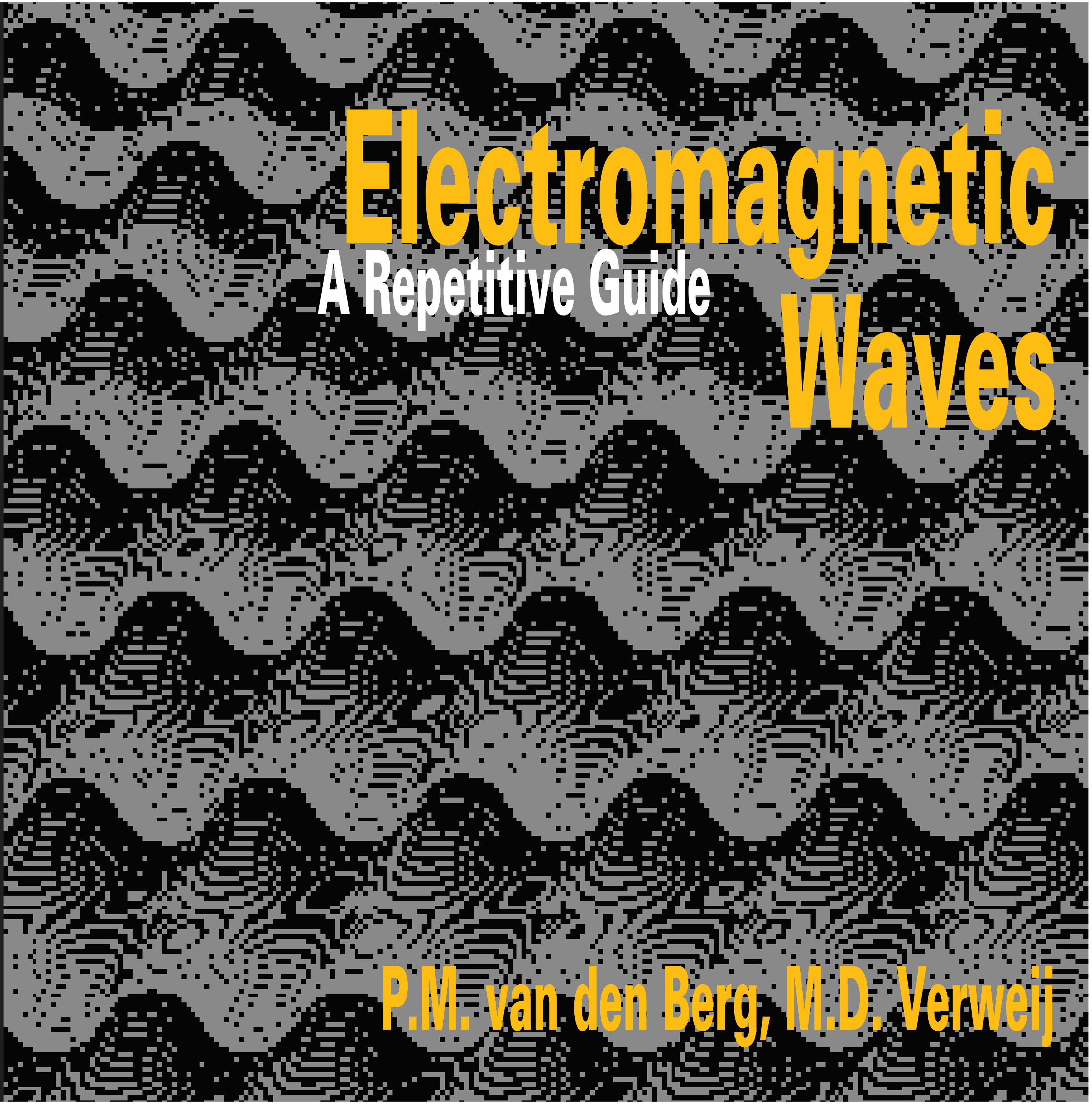
The aim of this course is to teach students to manipulate the fundamental formulas in order to solve a problem at hand. To focus on this skill and to overcome the problem of having to learn many formulas by heart, an outline of the main course book entitled **Electromagnetic Waves — An Introductory Course** (ISBN 90-407-1836-9) is presented in the current booklet.

Published by VSSD.

URL on this book:
http://www.vssd.nl/hlf/e016.htm

VS
SD

E016



Electromagnetic Waves

A Repetitive Guide

P.M. van den Berg, M.D. Verweij

Contents

1. Introduction (Cartesian Vectors)

A *vector* in a Cartesian reference frame is given by $\boldsymbol{x} = \underline{x} = x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3$.

The *addition/subtraction* of two vectors is given by

$$\boldsymbol{v} \pm \boldsymbol{w} = (v_1 \pm w_1)\boldsymbol{i}_1 + (v_2 \pm w_2)\boldsymbol{i}_2 + (v_3 \pm w_3)\boldsymbol{i}_3 \,.$$

The product of the scalar φ and the vector $\boldsymbol{v}$ is given by

$$\varphi\,\boldsymbol{v} = \varphi\, v_1\boldsymbol{i}_1 + \varphi\, v_2\boldsymbol{i}_2 + \varphi\, v_3\boldsymbol{i}_3 \,. \qquad \text{(vector)}$$

The scalar *dot product* of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by

$$\boldsymbol{v}\cdot\boldsymbol{w} = v_1\cdot w_1 + v_2\cdot w_2 + v_3\cdot w_3 = \boldsymbol{w}\cdot\boldsymbol{v} \,, \qquad \text{(scalar)}$$

while the vectoral *cross product* of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by

$$\boldsymbol{v}\times\boldsymbol{w} = (v_2w_3 - v_3w_2)\boldsymbol{i}_1 + (v_3w_1 - v_1w_3)\boldsymbol{i}_2 + (v_1w_2 - v_2w_1)\boldsymbol{i}_3 = -\boldsymbol{w}\times\boldsymbol{v} \,,$$

or in determinant notation

$$\boldsymbol{v}\times\boldsymbol{w} = \begin{vmatrix} \boldsymbol{i}_1 & \boldsymbol{i}_2 & \boldsymbol{i}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \,. \qquad \text{(vector)}$$

The *scalar triple product* of three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by

$$\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1) \,,$$

or in determinant notation

$$\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \,, \qquad \text{(scalar)}$$

with the properties

$$\begin{aligned} \boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) &= \boldsymbol{v}\cdot(\boldsymbol{w}\times\boldsymbol{u}) = \boldsymbol{w}\cdot(\boldsymbol{u}\times\boldsymbol{v}) \\ &= -\boldsymbol{w}\cdot(\boldsymbol{v}\times\boldsymbol{u}) = -\boldsymbol{v}\cdot(\boldsymbol{u}\times\boldsymbol{w}) = -\boldsymbol{u}\cdot(\boldsymbol{w}\times\boldsymbol{v}) \,. \end{aligned}$$

The *vectorial triple product* can be written as

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w} \,. \qquad \text{(vector)}$$

Differentiation with respect to a parameter t:

$$\begin{aligned}
\partial_t \boldsymbol{v} &= (\partial_t v_1)\boldsymbol{i}_1 + (\partial_t v_2)\boldsymbol{i}_2 + (\partial_t v_3)\boldsymbol{i}_3 \,, \\
\partial_t (\varphi \boldsymbol{v}) &= (\partial_t \varphi)\boldsymbol{v} + \varphi \partial_t \boldsymbol{v} \,, \\
\partial_t (\boldsymbol{v} \cdot \boldsymbol{w}) &= (\partial_t \boldsymbol{v}) \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \partial_t \boldsymbol{w} \,, \\
\partial_t (\boldsymbol{v} \times \boldsymbol{w}) &= (\partial_t \boldsymbol{v}) \times \boldsymbol{w} + \boldsymbol{v} \times \partial_t \boldsymbol{w} \,,
\end{aligned}$$

where $\partial_t = \frac{\partial}{\partial t}$.

Differentiation with respect to the spatial coordinates $\boldsymbol{x}$:

$$\begin{aligned}
\operatorname{grad} \varphi &= \boldsymbol{\nabla} \varphi \;=\; \partial_1 \varphi \boldsymbol{i}_1 + \partial_2 \varphi \boldsymbol{i}_2 + \partial_3 \varphi \boldsymbol{i}_3 \,, \\
\operatorname{div} \boldsymbol{v} &= \boldsymbol{\nabla} \cdot \boldsymbol{v} \;=\; \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 \,, \\
\operatorname{curl} \boldsymbol{v} &= \boldsymbol{\nabla} \times \boldsymbol{v} \;=\; (\partial_2 v_3 - \partial_3 v_2)\boldsymbol{i}_1 + (\partial_3 v_1 - \partial_1 v_3)\boldsymbol{i}_2 + (\partial_1 v_2 - \partial_2 v_1)\boldsymbol{i}_3 \,.
\end{aligned}$$

where $\boldsymbol{\nabla} = \boldsymbol{i}_1 \partial_1 + \boldsymbol{i}_2 \partial_2 + \boldsymbol{i}_3 \partial_3 = \boldsymbol{i}_1 \frac{\partial}{\partial x_1} + \boldsymbol{i}_2 \frac{\partial}{\partial x_2} + \boldsymbol{i}_3 \frac{\partial}{\partial x_3}$ is the so-called *nabla* or *del* operator satisfying the vector and partial differentiation rules, e.g.,

$$\begin{aligned}
\boldsymbol{\nabla}(\varphi + \psi) &= \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\psi \,, \\
\boldsymbol{\nabla} \cdot (\boldsymbol{v} + \boldsymbol{w}) &= \boldsymbol{\nabla} \cdot \boldsymbol{v} + \boldsymbol{\nabla} \cdot \boldsymbol{w} \,, \\
\boldsymbol{\nabla} \times (\boldsymbol{v} + \boldsymbol{w}) &= \boldsymbol{\nabla} \times \boldsymbol{v} + \boldsymbol{\nabla} \times \boldsymbol{w} \,, \\
\boldsymbol{\nabla}(\varphi \psi) &= (\boldsymbol{\nabla}\varphi)\, \psi + \varphi\, \boldsymbol{\nabla}\psi \,, \\
\boldsymbol{\nabla} \cdot (\varphi \boldsymbol{v}) &= (\boldsymbol{\nabla}\varphi) \cdot \boldsymbol{v} + \varphi\, \boldsymbol{\nabla} \cdot \boldsymbol{v} \,, \\
\boldsymbol{\nabla} \times (\varphi \boldsymbol{v}) &= (\boldsymbol{\nabla}\varphi) \times \boldsymbol{v} + \varphi\, \boldsymbol{\nabla} \times \boldsymbol{v} \,, \\
\boldsymbol{\nabla} \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot \boldsymbol{w} - \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{w}) \,, \\
\boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{w}) &= (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{w}\, \boldsymbol{\nabla} \cdot \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{w} + \boldsymbol{v}\, \boldsymbol{\nabla} \cdot \boldsymbol{w} \,, \\
\boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{w}) &= \boldsymbol{w} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) + (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{w}) + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{w} \,,
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\varphi) &= (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\varphi = (\partial_1^2 + \partial_2^2 + \partial_3^2)\varphi \,, \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\varphi) &= \boldsymbol{0} \,, \\
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) &= 0 \,, \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{v} \,.
\end{aligned}$$

The far-field approximation (→ Fig. 8.4):
Far away from the emitter, $r = (x_1^2 + x_3^2)^{\frac{1}{2}} \to \infty$, we have the far-field representation

$$\begin{aligned}\hat{E}_2 &\approx \frac{x_3}{r}\hat{e}_2^+(k\frac{x_1}{r}, j\omega)\left(\frac{k}{2\pi r}\right)^{\frac{1}{2}}\exp(-jkr + j\tfrac{1}{4}\pi)\\ &\approx -\left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}}\hat{h}_1^+(k\frac{x_1}{r}, j\omega)\left(\frac{k}{2\pi r}\right)^{\frac{1}{2}}\exp(-jkr + j\tfrac{1}{4}\pi)\,, \quad x_3 > 0.\end{aligned}$$

The amplitude is directly related to the spatial Fourier transform of the electric current through the emitter, by taking the quantity $k_1 = kx_1/r$ as the transform parameter.

The time-averaged power flow density in the far field is given by

$$\tfrac{1}{2}\mathrm{Re}[\hat{\boldsymbol{E}} \times \hat{\boldsymbol{H}}^*] \approx \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}}\frac{k}{4\pi r}\,|\hat{h}_1^+(k\frac{x_1}{r}, j\omega)|^2\,\frac{\boldsymbol{x}}{r}$$
$$\text{when } r = (x_1^2 + x_3^2)^{\frac{1}{2}} \to \infty\,.$$

To study the angular dependence of the far-field, the *directive gain* is introduced as the power flow in the observation direction, normalized to the angular-averaged power flow in the far field, viz.,

$$D(\theta) = \frac{\tfrac{1}{2}\mathrm{Re}\left[(\hat{\boldsymbol{E}} \times \hat{\boldsymbol{H}}^*)\cdot\dfrac{\boldsymbol{x}}{r}\right]}{\tfrac{1}{2}\mathrm{Re}\left[\dfrac{1}{2\pi}\displaystyle\int_{\theta=0}^{2\pi}(\hat{\boldsymbol{E}} \times \hat{\boldsymbol{H}}^*)\cdot\frac{\boldsymbol{x}}{r}\mathrm{d}\theta\right]}\,.$$

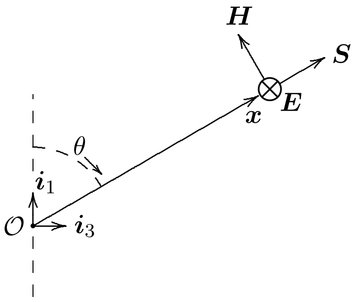


Figure 8.4. The orientation of the electromagnetic field vectors in the far field of an emitter with perpendicular electric current.

The *length* of a vector $\boldsymbol{v}$ is denoted as $|\boldsymbol{v}| = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} = (v_1^2 + v_2^2 + v_3^2)^{\frac{1}{2}}$. Rules for the spatial differentiation of a *function* $f = f(|\boldsymbol{x}|)$:

$$\begin{aligned}\boldsymbol{\nabla}|\boldsymbol{x}| &= \frac{\boldsymbol{x}}{|\boldsymbol{x}|}\,,\\ \boldsymbol{\nabla}|\boldsymbol{x}|^n &= n|\boldsymbol{x}|^{n-2}\boldsymbol{x}\,,\\ \boldsymbol{\nabla}f(|\boldsymbol{x}|) &= \partial f(|\boldsymbol{x}|)\,\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\,,\end{aligned}$$

where ∂f is the derivative of f with respect to its argument,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{x} &= 3\,,\\ \boldsymbol{\nabla}\times\boldsymbol{x} &= \boldsymbol{0}\,,\\ (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})|\boldsymbol{x}|^n &= n(n+1)|\boldsymbol{x}|^{n-2}\,.\end{aligned}$$

When $\boldsymbol{a}$ is a constant vector:

$$\begin{aligned}\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{x}) &= \boldsymbol{a}\,,\\ (\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{x} &= \boldsymbol{a}\,,\\ (\boldsymbol{a}\times\boldsymbol{\nabla})\times\boldsymbol{x} &= -2\boldsymbol{a}\,.\end{aligned}$$

Unit vector normal to a surface:
$\boldsymbol{\nabla}\varphi$ is oriented in a direction of the normal to the surface $\varphi =$ constant. Hence the normal vector $\boldsymbol{\nu}$ is given by

$$\boldsymbol{\nu} = \frac{\nabla\varphi}{|\nabla\varphi|}\,.$$

Gauss' integral theorem:

$$\begin{aligned}\oiint_{\boldsymbol{x}\in\partial\mathcal{D}}\boldsymbol{\nu}\cdot\boldsymbol{v}\,\mathrm{dA} &= \iiint_{\boldsymbol{x}\in\mathcal{D}}\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\mathrm{dV}\,,\\ \oiint_{\boldsymbol{x}\in\partial\mathcal{D}}\boldsymbol{\nu}\,\varphi\,\mathrm{dA} &= \iiint_{\boldsymbol{x}\in\mathcal{D}}\boldsymbol{\nabla}\varphi\,\mathrm{dV}\,,\\ \oiint_{\boldsymbol{x}\in\partial\mathcal{D}}\boldsymbol{\nu}\times\boldsymbol{w}\,\mathrm{dA} &= \iiint_{\boldsymbol{x}\in\mathcal{D}}\boldsymbol{\nabla}\times\boldsymbol{w}\,\mathrm{dV}\,,\end{aligned}$$

in which $\boldsymbol{\nu}$ is the unit vector normal to $\partial\mathcal{D}$ and oriented away from $\mathcal{D}$.

Stokes' integral theorem:

$$\begin{aligned}
\oint_{\boldsymbol{x}\in\partial\mathcal{S}} \boldsymbol{\tau}\cdot\boldsymbol{v}\,\mathrm{d}l &= \iint_{\boldsymbol{x}\in\mathcal{S}} (\boldsymbol{\nu}\times\boldsymbol{\nabla})\cdot\boldsymbol{v}\,\mathrm{dA} = \iint_{\boldsymbol{x}\in\mathcal{S}} \boldsymbol{\nu}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})\,\mathrm{dA}\,,\\
\oint_{\boldsymbol{x}\in\partial\mathcal{S}} \boldsymbol{\tau}\varphi\,\mathrm{d}l &= \iint_{\boldsymbol{x}\in\mathcal{S}} (\boldsymbol{\nu}\times\boldsymbol{\nabla})\varphi\,\mathrm{dA}\,,\\
\oint_{\boldsymbol{x}\in\partial\mathcal{S}} \boldsymbol{\tau}\times\boldsymbol{w}\,\mathrm{d}l &= \iint_{\boldsymbol{x}\in\mathcal{S}} (\boldsymbol{\nu}\times\boldsymbol{\nabla})\times\boldsymbol{w}\,\mathrm{dA}\,,
\end{aligned}$$

in which $\boldsymbol{\nu}$ is the unit vector normal to the surface area $\mathcal{S}$ and is oriented toward the side of advance of a right-hand screw as it is turned in the direction of the (unit) tangent vector $\boldsymbol{\tau}$ around $\partial\mathcal{S}$.

2. The Electromagnetic Field Equations

Maxwell's equations in vacuum:

$$\begin{aligned}
-\boldsymbol{\nabla}\times\boldsymbol{H}+\varepsilon_0\partial_t\boldsymbol{E} &= \boldsymbol{0}\,,\\
\boldsymbol{\nabla}\times\boldsymbol{E}+\mu_0\partial_t\boldsymbol{H} &= \boldsymbol{0}\,,
\end{aligned}$$

where

$\boldsymbol{E}$ = electric field strength (V/m),
$\boldsymbol{H}$ = magnetic field strength (A/m),
ε_0 = permittivity in vacuum (8.8541878×10^{-12} F/m),
μ_0 = permeability in vacuum ($4\pi\times10^{-7}$ H/m).

Maxwell's equations in matter:

$$\begin{aligned}
-\boldsymbol{\nabla}\times\boldsymbol{H}+\boldsymbol{J}+\partial_t\boldsymbol{D} &= -\boldsymbol{J}^{ext}\,,\\
\boldsymbol{\nabla}\times\boldsymbol{E}+\partial_t\boldsymbol{B} &= -\boldsymbol{K}^{ext}\,,
\end{aligned}$$

where

$\boldsymbol{J}$ = volume density of electric current (A/m^2),
$\boldsymbol{D}$ = electric flux density (C/m^2),
$\boldsymbol{B}$ = magnetic flux density (T),
$\boldsymbol{J}^{ext}$ = volume density of material electric current (A/m^2),
$\boldsymbol{K}^{ext}$ = volume density of material magnetic current (V/m^2).

The sheet emitter with a perpendicular electric current (→ Fig. 8.3):
The impressed electric current density is

$$\hat{J}_1^{ext}=0\,,\quad \hat{J}_2^{ext}=\begin{cases}\hat{I}_\Delta(x_1,s)\delta(x_3)\,, & |x_3|<\frac{1}{2}a\,,\\ 0\,, & |x_3|>\frac{1}{2}a\,,\end{cases}\quad \hat{J}_3^{ext}=0\,,$$

where $\hat{I}_\Delta$ (in A/m) is the electric current per unit length (of the x_1-direction). The non-zero electromagnetic field components are $\hat{E}_2$, $\hat{H}_1=(j\omega\mu)^{-1}\partial_3\hat{E}_2$, $\hat{H}_3=-(j\omega\mu)^{-1}\partial_1\hat{E}_2$. The electromagnetic field may be written as an infinite superposition of plane-wave constituents:

$$\hat{E}_2(x_1,x_3,j\omega) = \frac{1}{2\pi}\int_{k_1=-\infty}^{\infty}\hat{e}_2^{+}(k_1,j\omega)\exp(-jk_1x_1-jk_3x_3)\mathrm{d}k_1\,,\quad x_3>0,$$

where the amplitude

$$\hat{e}_2^{+}=-\frac{\omega\mu}{k_3}\hat{h}_1^{+}$$

follows from a Fourier transform of the impressed electric current through

$$\hat{h}_1^{+}(k_1,j\omega)=\int_{x_1=-\frac{1}{2}a}^{\frac{1}{2}a}\tfrac{1}{2}\hat{I}_\Delta(x_1,j\omega)\exp(jk_1x_1)\,\mathrm{d}x_1\,.$$

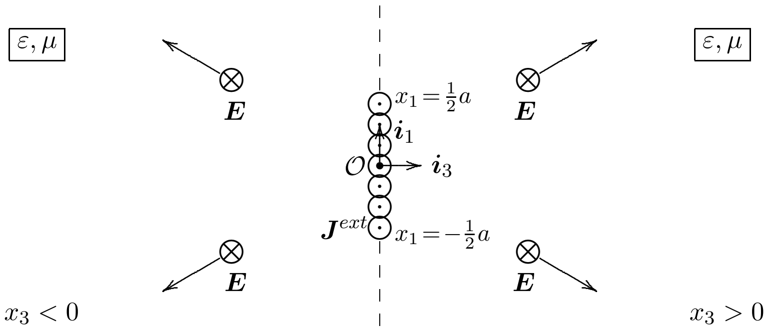


Figure 8.3. Electric-current sheet as an emitter of perpendicularly polarized electromagnetic waves.

and where the amplitude

$$\hat{h}_2^+(k_1, j\omega) = \int_{x_1=-\frac{1}{2}a}^{\frac{1}{2}a} \tfrac{1}{2}\hat{I}_\Delta(x_1, j\omega)\exp(jk_1x_1)\,\mathrm{d}x_1$$

follows from a Fourier transform of the impressed electric current.

The far-field approximation (→ Fig. 8.2):
Far away from the emitter, $r = (x_1^2 + x_3^2)^{\frac{1}{2}} \to \infty$, we have the far-field representation

$$\hat{H}_2 \approx \frac{x_3}{r}\hat{h}_2^+(k\frac{x_1}{r}, j\omega)\left(\frac{k}{2\pi r}\right)^{\frac{1}{2}}\exp(-jkr + j\tfrac{1}{4}\pi)\,, \quad x_3 > 0\,.$$

The amplitude is directly related to the spatial Fourier transform of the electric current through the emitter, by taking the quantity $k_1 = kx_1/r$ as the transform parameter.

The time-averaged power flow density in the far field is given by

$$\tfrac{1}{2}\mathrm{Re}[\hat{\boldsymbol{E}} \times \hat{\boldsymbol{H}}^*] \approx \left(\frac{x_3}{r}\right)^2\left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}}\frac{k}{4\pi r}\,|\hat{h}_2^+(k\frac{x_1}{r}, j\omega)|^2\,\frac{\boldsymbol{x}}{r}\,,$$
$$\text{when } r = (x_1^2 + x_3^2)^{\frac{1}{2}} \to \infty\,.$$

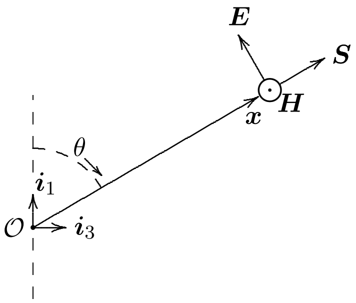


Figure 8.2. The orientation of the electromagnetic field vectors in the far field of an emitter with parallel electric current.

Constitutive relations:

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{x},t) &= \sigma(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x},t)\,,\\ \boldsymbol{D}(\boldsymbol{x},t) &= \varepsilon(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x},t)\,,\\ \boldsymbol{B}(\boldsymbol{x},t) &= \mu(\boldsymbol{x})\boldsymbol{H}(\boldsymbol{x},t)\,,\end{aligned}$$

where

σ = conductivity (S/m),
ε = permittivity (F/m),
μ = permeability (H/m).

In *vacuum* we have $\sigma = 0$, $\varepsilon = \varepsilon_0 = \frac{1}{\mu_0 c_0^2}$, $\mu = \mu_0$.
Customarily:

$\varepsilon_r = \varepsilon/\varepsilon_0$ = relative permittivity,
$\mu_r = \mu/\mu_0$ = relative permeability.

Field equations:

$$\begin{aligned}-\boldsymbol{\nabla} \times \boldsymbol{H} + \sigma\boldsymbol{E} + \varepsilon\partial_t\boldsymbol{E} &= -\boldsymbol{J}^{ext}\,,\\ \boldsymbol{\nabla} \times \boldsymbol{E} + \mu\partial_t\boldsymbol{H} &= -\boldsymbol{K}^{ext}\,,\end{aligned}$$

or

$$\begin{aligned}-(\partial_2H_3 - \partial_3H_2) + \sigma E_1 + \varepsilon\partial_tE_1 &= -J_1^{ext}\,,\\ -(\partial_3H_1 - \partial_1H_3) + \sigma E_2 + \varepsilon\partial_tE_2 &= -J_2^{ext}\,,\\ -(\partial_1H_2 - \partial_2H_1) + \sigma E_3 + \varepsilon\partial_tE_3 &= -J_3^{ext}\,,\end{aligned}$$

$$\begin{aligned}\partial_2E_3 - \partial_3E_2 + \mu\partial_tH_1 &= -K_1^{ext}\,,\\ \partial_3E_1 - \partial_1E_3 + \mu\partial_tH_2 &= -K_2^{ext}\,,\\ \partial_1E_2 - \partial_2E_1 + \mu\partial_tH_3 &= -K_3^{ext}\,.\end{aligned}$$

The *Poynting vector* $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ (in W/m^2) quantifies the amount of electromagnetic power flow per unit area; the components are

$$S_1 = E_2H_3 - E_3H_2\,,\;\; S_2 = E_3H_1 - E_1H_3\,,\;\; S_3 = E_1H_2 - E_2H_1\,.$$

Boundary conditions at an interface $\partial\mathcal{D}$ between different media:

$\boldsymbol{\nu} \times \boldsymbol{H}$ is continuous across $\partial\mathcal{D}$,
$\boldsymbol{\nu} \times \boldsymbol{E}$ is continuous across $\partial\mathcal{D}$.

Frequency-domain representations for causal fields:

Let $f = f(\boldsymbol{x}, t)$ denote a causal ($f = 0$ when $t < 0$) field quantity. Then,

$$\hat{f}(\boldsymbol{x}, s) = \int_{t=0}^{\infty} \exp(-st) f(\boldsymbol{x}, t) \mathrm{d}t \quad \text{for } \mathrm{Re}(s) > s_0 \, ,$$

where s is the complex Laplace transform parameter.

Field equations:
$$\begin{aligned} -\boldsymbol{\nabla} \times \hat{\boldsymbol{H}} + (\sigma + s\varepsilon)\hat{\boldsymbol{E}} &= -\hat{\boldsymbol{J}}^{ext} \, , \\ \boldsymbol{\nabla} \times \hat{\boldsymbol{E}} + s\mu\hat{\boldsymbol{H}} &= -\hat{\boldsymbol{K}}^{ext} \, , \end{aligned}$$

or

$$\begin{aligned} -(\partial_2 \hat{H}_3 - \partial_3 \hat{H}_2) + (\sigma + s\varepsilon)\hat{E}_1 &= -\hat{J}_1^{ext} \, , \\ -(\partial_3 \hat{H}_1 - \partial_1 \hat{H}_3) + (\sigma + s\varepsilon)\hat{E}_2 &= -\hat{J}_2^{ext} \, , \\ -(\partial_1 \hat{H}_2 - \partial_2 \hat{H}_1) + (\sigma + s\varepsilon)\hat{E}_3 &= -\hat{J}_3^{ext} \, , \end{aligned}$$

$$\begin{aligned} \partial_2 \hat{E}_3 - \partial_3 \hat{E}_2 + s\mu\hat{H}_1 &= -\hat{K}_1^{ext} \, , \\ \partial_3 \hat{E}_1 - \partial_1 \hat{E}_3 + s\mu\hat{H}_2 &= -\hat{K}_2^{ext} \, , \\ \partial_1 \hat{E}_2 - \partial_2 \hat{E}_1 + s\mu\hat{H}_3 &= -\hat{K}_3^{ext} \, . \end{aligned}$$

Frequency-domain analysis is arrived at by taking $s \to j\omega$, via $\mathrm{Re}(s) > 0$, where ω is the (real) *angular frequency*. Then, either, for *causal states*,

$$\begin{aligned} \hat{f}(\boldsymbol{x}, j\omega) &= \int_{t=0}^{\infty} \exp(-j\omega t) f(\boldsymbol{x}, t) \mathrm{d}t \quad \text{for all real } \omega \, , \\ f(\boldsymbol{x}, t) &= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \exp(j\omega t) \hat{f}(\boldsymbol{x}, j\omega) \mathrm{d}\omega \quad \text{for all } t \, , \end{aligned}$$

or, for *steady-states*,

$$f(\boldsymbol{x}, t) = \mathrm{Re}\left[\hat{f}(\boldsymbol{x}, j\omega) \exp(j\omega t)\right] \, .$$

In the complex frequency domain, the *length* of a complex vector $\boldsymbol{v}$ is denoted as $|\boldsymbol{v}| = (\boldsymbol{v} \cdot \boldsymbol{v}^*)^{\frac{1}{2}} = (v_1 v_1^* + v_2 v_2^* + v_3 v_3^*)^{\frac{1}{2}}$. The asterisk denotes complex conjugate.

Polarization state:
In general, the electric field strength is elliptically polarized.
– It is *linearly polarized* when $\hat{\boldsymbol{E}}(\boldsymbol{x}, j\omega) \times \hat{\boldsymbol{E}}^*(\boldsymbol{x}, j\omega) = \boldsymbol{0}$.
– It is *circularly polarized* when $\hat{\boldsymbol{E}}(\boldsymbol{x}, j\omega) \cdot \hat{\boldsymbol{E}}(\boldsymbol{x}, j\omega) = 0$.

---

8. Excitation of Two-dimensional Electromagnetic Waves

We only consider the steady-state ($s = j\omega$, $\omega \geq 0$). The medium is homogeneous and lossless ($\sigma = 0$). Further, the emitter carries no magnetic current ($\boldsymbol{K}^{ext} = \boldsymbol{0}$).

The sheet emitter with a parallel electric current (→ Fig. 8.1):
The impressed electric current density is

$$\hat{J}_1^{ext} = \begin{cases} -\hat{I}_\Delta(x_1, s)\delta(x_3) \, , & |x_3| < \frac{1}{2}a \, , \\ 0 \, , & |x_3| > \frac{1}{2}a \, , \end{cases} \qquad \hat{J}_2^{ext} = 0 \, , \quad \hat{J}_3^{ext} = 0 \, ,$$

where $\hat{I}_\Delta$ (in A/m) is the electric current per unit length (of the x_2-direction). The non-zero electromagnetic field components are $\hat{H}_2$, $\hat{E}_1 = -(j\omega\varepsilon)^{-1}\partial_3\hat{H}_2$, $\hat{E}_3 = (j\omega\varepsilon)^{-1}\partial_1\hat{H}_2$. The electromagnetic field may be written as an infinite superposition of plane-wave constituents: (we consider only $x_3 > 0$)

$$\hat{H}_2(x_1, x_3, j\omega) = \frac{1}{2\pi} \int_{k_1=-\infty}^{\infty} \hat{h}_2^+(k_1, j\omega) \exp(-jk_1 x_1 - jk_3 x_3) \mathrm{d}k_1 \, , \quad x_3 > 0,$$

where

$$k_3 = \begin{cases} (\omega^2 \varepsilon\mu - k_1^2)^{\frac{1}{2}} \, , & |k_1| \leq \omega(\varepsilon\mu)^{\frac{1}{2}} \, , \\ -j(k_1^2 - \omega^2 \varepsilon\mu)^{\frac{1}{2}} \, , & |k_1| > \omega(\varepsilon\mu)^{\frac{1}{2}} \, , \end{cases}$$

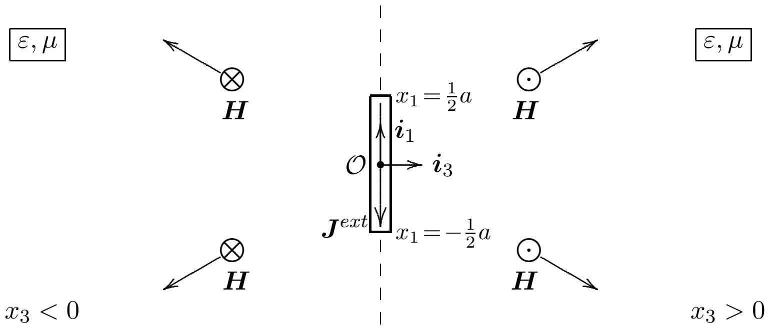


Figure 8.1. Electric-current sheet as an emitter of parallelly polarized electromagnetic waves.

For a TM$_m$-mode, the non-zero electromagnetic field components are $\hat{H}_{2;m}$, $\hat{E}_{1;m} = -(j\omega\varepsilon)^{-1}\partial_3\hat{H}_{2;m}$, $\hat{E}_{3;m} = (j\omega\varepsilon)^{-1}\partial_1\hat{H}_{2;m}$, where

$$\hat{H}_{2;m} = \frac{\hat{A}_m^{(TM)}}{\cos(\psi_m)}\begin{cases} \cos(\kappa_m^{(2)}a-\psi_m)\exp[-\kappa_m^{(3)}(x_1-a)]\exp(-jk_{3;m}x_3) \\ \qquad\qquad\qquad\text{when } a < x_1 < \infty\,, \\ \cos(\kappa_m^{(2)}x_1-\psi_m)\exp(-jk_{3;m}x_3) \\ \qquad\qquad\qquad\text{when } 0 < x_1 < a\,, \\ \cos(\psi_m)\exp(\kappa_m^{(1)}x_1)\exp(-jk_{3;m}x_3) \\ \qquad\qquad\qquad\text{when } -\infty < x_1 < 0\,, \end{cases}$$

in which $\psi_m = \arctan\left(\frac{\varepsilon^{(2)}\,\kappa_m^{(1)}}{\varepsilon^{(1)}\,\kappa_m^{(2)}}\right)$. The quantities $\kappa_m^{(i)}$, $i = 1,2,3$, are related to the propagation coefficient $k_{3;m}$ through the *dispersion equation*:

$$\kappa_m^{(2)}a = \arctan\left(\frac{\varepsilon^{(2)}\,\kappa_m^{(1)}}{\varepsilon^{(1)}\,\kappa_m^{(2)}}\right) + \arctan\left(\frac{\varepsilon^{(2)}\,\kappa_m^{(3)}}{\varepsilon^{(3)}\,\kappa_m^{(2)}}\right) + m\pi\,.$$

In both the TE-case and the TM-case, $\kappa_m^{(i)}$, $i = 1,2,3$, is defined as

$$\kappa_m^{(1)} = [k_{3;m}^2 - \omega^2\varepsilon^{(1)}\mu_0]^{\frac{1}{2}},\ \kappa_m^{(2)} = [\omega^2\varepsilon^{(2)}\mu_0 - k_{3;m}^2]^{\frac{1}{2}},\ \kappa_m^{(3)} = [k_{3;m}^2 - \omega^2\varepsilon^{(3)}\mu_0]^{\frac{1}{2}}.$$

With these definitions (and $[\ \ ]^{\frac{1}{2}} > 0$), the solutions of the dispersion equation, $k_{3;m}(\omega)$, $m = 0,1,2,\cdots,\infty$, form an innumerable set of real and complex numbers. The real values form a finite set with

$$\max[k^{(1)},k^{(3)}] < k_{3;m} < k^{(2)}\,, \qquad k^{(i)} = \omega(\varepsilon^{(i)}\mu_0)^{\frac{1}{2}}\,,\ i = 1,2,3\,,$$

and each value $k_{3;m}$ is the *propagation constant* of the m-th *guided mode* of the dielectric slab waveguide.

Often, the *effective index of refraction* (mode index) is introduced as

$$n_{\mathit{eff},m} = k_{3;m}/k_0$$

where k_0 is the free-space wavenumber ($k_0 = 2\pi/\lambda_0 = \omega/c_0$). The effective index of a mode is located in $\max[n^{(1)},n^{(3)}] < n_{\mathit{eff},m} < n^{(2)}$. Assuming $n_1 > n_3$, the cut-off frequency of a guided mode follows from $\kappa_m^{(1)}(\omega) = 0$.

Phase and group velocity:

In waveguides the phase velocity of a mode $v_{\phi,m} = \omega/k_{3;m}$ differs from the group velocity $v_{g,m} = 1/(\partial_\omega k_{3;m})$.

Poynting's theorem:

Power balance in differential form:

$$\boldsymbol{\nabla}\cdot\boldsymbol{S} + \dot{w}^h + \partial_t(w^e + w^m) = -\boldsymbol{E}\cdot\boldsymbol{J}^{ext} - \boldsymbol{H}\cdot\boldsymbol{K}^{ext}\,,$$

where

- $\dot{w}^h = \sigma\boldsymbol{E}\cdot\boldsymbol{E}$ is the volume density of electromagnetic power that is irreversibly dissipated into heat,
- $w^e = \frac{1}{2}\varepsilon\boldsymbol{E}\cdot\boldsymbol{E}$ is the volume density of reversibly stored electric field energy,
- $w^m = \frac{1}{2}\mu\boldsymbol{H}\cdot\boldsymbol{H}$ is the volume density of reversibly stored magnetic field energy,
- $-\boldsymbol{E}\cdot\boldsymbol{J}^{ext} - \boldsymbol{H}\cdot\boldsymbol{K}^{ext}$ represents the electromagnetic power that is generated by the electromagnetic sources.

For *steady-states* the time average Poynting's vector is introduced as

$$\langle\boldsymbol{S}\rangle_T = \tfrac{1}{2}\,\mathrm{Re}\left[\hat{\boldsymbol{E}}\times\hat{\boldsymbol{H}}^*\right]\,,$$

and the complex form of Poynting's theorem is given by

$$\boldsymbol{\nabla}\cdot(\hat{\boldsymbol{E}}\times\hat{\boldsymbol{H}}^*) + (\sigma - j\omega\varepsilon)\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{E}}^* + j\omega\mu\hat{\boldsymbol{H}}\cdot\hat{\boldsymbol{H}}^* = -\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{J}}^{ext*} - \hat{\boldsymbol{H}}^*\cdot\hat{\boldsymbol{K}}^{ext}\,,$$

where the term $\frac{1}{2}\sigma\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{E}}^*$ represents the time average of the volume density of heat dissipated by the electromagnetic field.

Equations in integral form:

Maxwell's equations (see Stokes' integral theorem):

$$\begin{aligned} -\oint_{\boldsymbol{x}\in\partial\mathcal{S}}\boldsymbol{\tau}\cdot\boldsymbol{H}\mathrm{d}l + \iint_{\boldsymbol{x}\in\mathcal{S}}\boldsymbol{\nu}\cdot[\sigma\boldsymbol{E}+\varepsilon\partial_t\boldsymbol{E}]\mathrm{dA} &= -\iint_{\boldsymbol{x}\in\mathcal{S}}\boldsymbol{\nu}\cdot\boldsymbol{J}^{ext}\mathrm{dA}\,, \\ \oint_{\boldsymbol{x}\in\partial\mathcal{S}}\boldsymbol{\tau}\cdot\boldsymbol{E}\mathrm{d}l + \iint_{\boldsymbol{x}\in\mathcal{S}}\mu\boldsymbol{\nu}\cdot\partial_t\boldsymbol{H}\mathrm{dA} &= -\iint_{\boldsymbol{x}\in\mathcal{S}}\boldsymbol{\nu}\cdot\boldsymbol{K}^{ext}\mathrm{dA}\,. \end{aligned}$$

Poynting's theorem (see Gauss' integral theorem):

$$\begin{aligned} \iint_{\boldsymbol{x}\in\partial\mathcal{D}}\boldsymbol{\nu}\cdot\boldsymbol{S}\mathrm{dA} + \iiint_{\boldsymbol{x}\in\mathcal{D}}\sigma\boldsymbol{E}\cdot\boldsymbol{E}\mathrm{dV} + \iiint_{\boldsymbol{x}\in\mathcal{D}}\partial_t[\tfrac{1}{2}\varepsilon\boldsymbol{E}\cdot\boldsymbol{E}+\tfrac{1}{2}\mu\boldsymbol{H}\cdot\boldsymbol{H}]\mathrm{dV} \\ = -\iiint_{\boldsymbol{x}\in\mathcal{D}}(\boldsymbol{E}\cdot\boldsymbol{J}^{ext}+\boldsymbol{H}\cdot\boldsymbol{K}^{ext})\mathrm{dV}\,. \end{aligned}$$

3. One-dimensional Electromagnetic Waves

Excitation (→ Fig. 3.1):

A planar electric-current sheet emits a plane wave in the positive x_3-direction with the field components

$$\begin{aligned} \hat{E}_1 &= \tfrac{1}{2} Z \hat{I}_\Delta \exp(-\gamma x_3) \quad \text{for } x_3 > 0 \,, \\ \hat{H}_2 &= \tfrac{1}{2} \hat{I}_\Delta \exp(-\gamma x_3) \quad \text{for } x_3 > 0 \,. \end{aligned}$$

where $\gamma = [(\sigma + s\varepsilon)s\mu]^{\frac{1}{2}}$ is the *propagation coefficient*, with $\mathrm{Re}(\gamma) \geq 0$. Further, the wave impedance Z is given by

$$Z = \left(\frac{s\mu}{\sigma + s\varepsilon} \right)^{\frac{1}{2}} .$$

Sometimes the *wave admittance* is introduced as $Y = Z^{-1}$.

In case of a *steady-state*, the exciting current is written as

$$I_\Delta(t) = \mathrm{Re}\left[\hat{I}_\Delta(j\omega) \exp(j\omega t) \right]$$

and, with $s = j\omega$, the propagation coefficient is written as

$$\gamma(j\omega) = \alpha(\omega) + j\beta(\omega) \,,$$

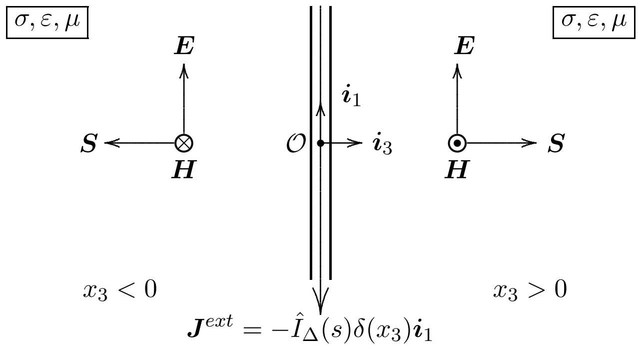


Figure 3.1. Electric-current sheet emitter.

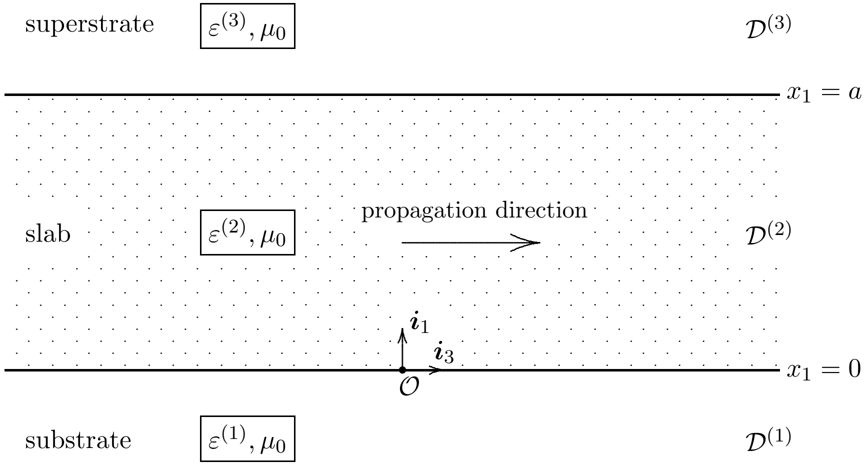


Figure 7.2. A dielectric slab waveguide.

Dielectric slab waveguide (→ Fig. 7.2):

For a TE$_m$-mode, the non-zero electromagnetic field components are $\hat{E}_{2;m}$, $\hat{H}_{1;m} = (j\omega\mu)^{-1}\partial_3 \hat{E}_{2;m}$, $\hat{H}_{3;m} = -(j\omega\mu)^{-1}\partial_1 \hat{E}_{2;m}$, where

$$\hat{E}_{2;m} = \frac{\hat{A}_m^{(TE)}}{\cos(\psi_m)} \begin{cases} \cos(\kappa_m^{(2)} a - \psi_m) \exp[-\kappa_m^{(3)}(x_1 - a)] \exp(-jk_{3;m}x_3) & \text{when } a < x_1 < \infty \,, \\ \cos(\kappa_m^{(2)} x_1 - \psi_m) \exp(-jk_{3;m}x_3) & \text{when } 0 < x_1 < a \,, \\ \cos(\psi_m) \exp(\kappa_m^{(1)} x_1) \exp(-jk_{3;m}x_3) & \text{when } -\infty < x_1 < 0 \,, \end{cases}$$

in which $\psi_m = \arctan\left(\frac{\kappa_m^{(1)}}{\kappa_m^{(2)}}\right)$. The quantities $\kappa_m^{(i)}$, $i = 1, 2, 3$, are related to the propagation coefficient $k_{3;m}$ through the *dispersion equation*:

$$\kappa_m^{(2)} a = \arctan\left(\frac{\kappa_m^{(1)}}{\kappa_m^{(2)}}\right) + \arctan\left(\frac{\kappa_m^{(3)}}{\kappa_m^{(2)}}\right) + m\pi \,.$$

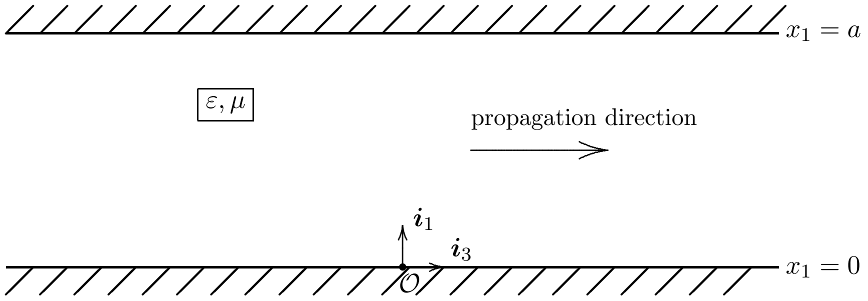


Figure 7.1. A parallel-plate waveguide.

Parallel-plate waveguide (→ Fig. 7.1):

In the TE-case the non-zero electromagnetic field components are $\hat{E}_2$, $\hat{H}_1 = (j\omega\mu)^{-1}\partial_3\hat{E}_2$, $\hat{H}_3 = -(j\omega\mu)^{-1}\partial_1\hat{E}_2$, where

$$\hat{E}_2(x_1, x_3, j\omega) = 2j\sum_{m=1}^{\infty}\hat{A}_m^{(TE)}\sin(\frac{m\pi}{a}x_1)\exp(-jk_{3;m}x_3) \, .$$

In the TM-case the non-zero electromagnetic field components are $\hat{H}_2$, $\hat{E}_1 = -(j\omega\varepsilon)^{-1}\partial_3\hat{H}_2$, $\hat{E}_3 = (j\omega\varepsilon)^{-1}\partial_1\hat{H}_2$, where

$$\hat{H}_2(x_1, x_3, j\omega) = 2\sum_{m=0}^{\infty}\hat{A}_m^{(TM)}\cos(\frac{m\pi}{a}x_1)\exp(-jk_{3;m}x_3) \, .$$

The *propagation constants* are

$$k_{3;m} = [k^2 - (\frac{m\pi}{a})^2]^{\frac{1}{2}} \text{ for } k \geq \frac{m\pi}{a} \, , \quad k_{3;m} = -j[(\frac{m\pi}{a})^2 - k^2]^{\frac{1}{2}} \text{ for } k < \frac{m\pi}{a} \, ,$$

where $k = \omega(\varepsilon\mu)^{\frac{1}{2}}$ is the wavenumber. For $m \leq ka/\pi$, we have a propagating mode, while for $ka/\pi < m$, we have a non-propagating mode. Only for angular frequencies larger than the *cut-off angular frequency*,

$$\omega_{c,m} = \frac{m\pi}{a}(\varepsilon\mu)^{-\frac{1}{2}} \, ,$$

the m-th mode is propagating.

where α is the *attenuation coefficient* and β is the *phase coefficient*, given by

$$\alpha = \omega(\varepsilon\mu)^{\frac{1}{2}}\left\{\frac{1}{2}\left[\left(\frac{\sigma}{\omega\varepsilon}\right)^2 + 1\right]^{\frac{1}{2}} - \frac{1}{2}\right\}^{\frac{1}{2}} \, ,$$

$$\beta = \omega(\varepsilon\mu)^{\frac{1}{2}}\left\{\frac{1}{2}\left[\left(\frac{\sigma}{\omega\varepsilon}\right)^2 + 1\right]^{\frac{1}{2}} + \frac{1}{2}\right\}^{\frac{1}{2}} \, .$$

For highly conductive media ($\sigma \gg \omega\varepsilon$), we define the *skin depth* as

$$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{\frac{1}{2}} \, .$$

For lossless media ($\sigma = 0$), the *wave speed* is obtained as $c = (\varepsilon\mu)^{-\frac{1}{2}}$, while the *wavelength* is given by

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega(\varepsilon\mu)^{\frac{1}{2}}} = \frac{c}{f} \, .$$

The time-domain *transient wavefield* in a lossless medium is obtained as

$$E_1(x_3, t) = \tfrac{1}{2}ZI_\Delta(t - \frac{x_3}{c}) \quad \text{for } x_3 > 0 \, ,$$
$$H_2(x_3, t) = \tfrac{1}{2}I_\Delta(t - \frac{x_3}{c}) \quad \text{for } x_3 > 0 \, ,$$

which is a one-dimensional wave that propagates in the direction of increasing x_3, i.e., away from the generating source, with the wave speed c with the pulse shape of the generating transient electric current.

Reflection and transmission (→ Fig. 3.2):

The *incident wavefield* propagates in medium (1) from the emitter in the positive x_3-direction and is given by

$$\hat{E}_1^i = \hat{e}_1^i\exp(-\gamma^{(1)}x_3) \, ,$$
$$\hat{H}_2^i = Y^{(1)}\hat{e}_1^i\exp(-\gamma^{(1)}x_3) \, .$$

The *reflected wavefield* propagates in medium (1) from the interface in the negative x_3-direction and is given by

$$\hat{E}_1^r = R_\perp\,\hat{e}_1^i\exp(\gamma^{(1)}x_3) \, ,$$
$$\hat{H}_2^r = -Y^{(1)}R_\perp\,\hat{e}_1^i\exp(\gamma^{(1)}x_3) \, .$$

The *transmitted wavefield* propagates in medium (2) from the interface in the positive x_3-direction and is given by

$$\begin{aligned}\hat{E}_1^t &= T_\perp\,\hat{e}_1^i \exp(-\gamma^{(2)}x_3)\,,\\ \hat{H}_2^t &= Y^{(2)}T_\perp\,\hat{e}_1^i \exp(-\gamma^{(2)}x_3)\,.\end{aligned}$$

The propagation coefficients in the two media are

$$\gamma^{(1)} = \left[(\sigma^{(1)} + s\varepsilon^{(1)})s\mu^{(1)}\right]^{\frac{1}{2}}\ ,\quad \gamma^{(2)} = \left[(\sigma^{(2)} + s\varepsilon^{(2)})s\mu^{(2)}\right]^{\frac{1}{2}}\ ,$$

with $\mathrm{Re}[\gamma^{(1)}] \geq 0$ and $\mathrm{Re}[\gamma^{(2)}] \geq 0$, while the wave admittances in the two media are

$$Y^{(1)} = \left(\frac{\sigma^{(1)} + s\varepsilon^{(1)}}{s\mu^{(1)}}\right)^{\frac{1}{2}}\ ,\quad Y^{(2)} = \left(\frac{\sigma^{(2)} + s\varepsilon^{(2)}}{s\mu^{(2)}}\right)^{\frac{1}{2}}\ .$$

The reflection and transmission coefficients follow from the application of the boundary conditions at the interface $x_3 = 0$ and are given by

$$R_\perp = \frac{Y^{(1)} - Y^{(2)}}{Y^{(1)} + Y^{(2)}}\ ,\quad T_\perp = \frac{2Y^{(1)}}{Y^{(1)} + Y^{(2)}}\ .$$

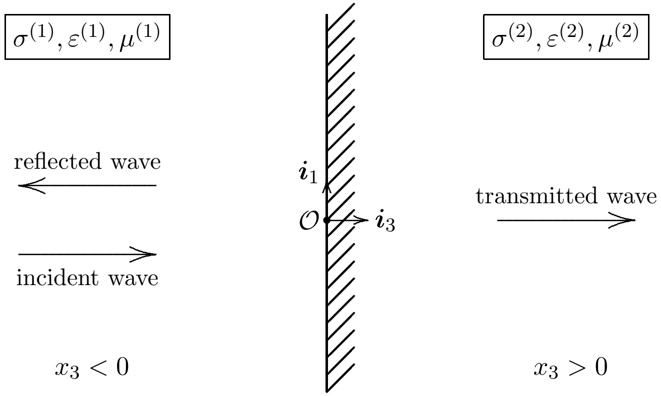


Figure 3.2. Reflection by and transmission through a plane interface.

where

$$\Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0}$$

is the reflection coefficient of the wave reflected at the source location $x_3 = 0$.

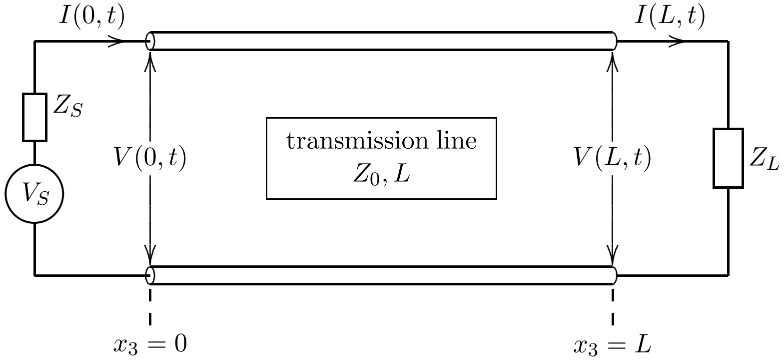


Figure 6.3. An impulsive source connected to the transmission line.

7. Electromagnetic Waveguides

Let us assume that $\boldsymbol{i}_3$ is the direction of propagation. We only consider the steady-state: ($s = j\omega$, $\omega \geq 0$). The behaviour of the waves in a waveguide is predicted by the homogeneous Maxwell equations and the boundary conditions at the waveguide surfaces. We distinguish the following solutions:

- Transverse Electric (TE) modes: The electric field is perpendicular to the propagation direction, $\hat{E}_3 = 0$.
- Transverse Magnetic (TM) modes: The magnetic field is perpendicular to the propagation direction, $\hat{H}_3 = 0$.
- Transverse Electromagnetic (TEM) modes: Both the electric and the magnetic field are perpendicular to the propagation direction, $\hat{E}_3 = \hat{H}_3 = 0$ (see Chapter 6).

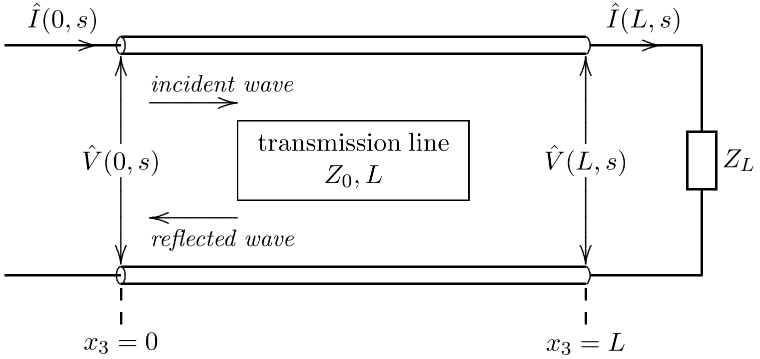


Figure 6.2. The terminated transmission line.

Lossless transmission line, steady-state analysis:

In the lossless case ($\sigma = 0$) and steady-state ($s = j\omega$, $\omega \geq 0$) we have $\gamma = jk$, with wave number $k = \omega/c$ and the wave speed $c = (\varepsilon\mu)^{-\frac{1}{2}}$. Then, the input impedance is given by

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(kL)}{Z_0 + jZ_L \tan(kL)} \, .$$

Further, the *time average of the power* transmitted through the cross-section of the transmission line is obtained as

$$\tfrac{1}{2}\mathrm{Re}\left[\iint_{(x_1,x_2)\in\mathcal{D}} (\hat{\boldsymbol{E}} \times \hat{\boldsymbol{H}}^*) \cdot \boldsymbol{i}_3 \, \mathrm{dA}\right] = \tfrac{1}{2}\mathrm{Re}\left[\hat{V}\hat{I}^*\right] \, .$$

Lossless transmission line, transient wavefield (→ Fig. 6.3):

In the lossless case the time-domain wavefield is given by

$$\begin{aligned} V(x_3,t) &= v^+(t - \frac{x_3}{c}) + \Gamma_L v^+(t - \frac{2L - x_3}{c}) \, , \\ I(x_3,t) &= Y_0 \left[v^+(t - \frac{x_3}{c}) - \Gamma_L v^+(t - \frac{2L - x_3}{c}) \right] \, . \end{aligned}$$

with

$$v^+(t) = \frac{Z_0}{Z_0 + Z_S} \left[\sum_{n=0}^{\infty} (\Gamma_S \Gamma_L)^n V_S(t - \frac{2nL}{c}) \right] \, ,$$

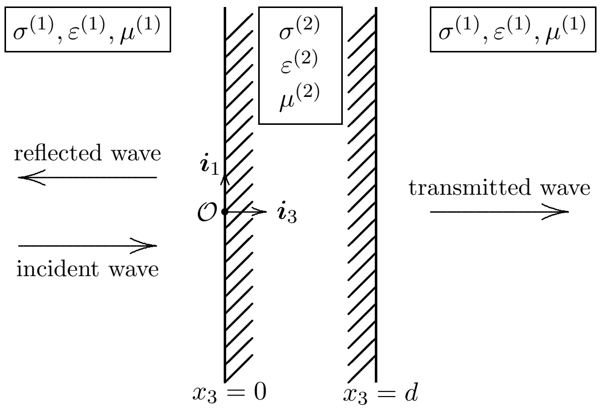


Figure 3.3. Reflection by and transmission through a shielding plate.

Shielding (→ Fig. 3.3):

The electric-field components of the incident wavefield, the reflected wavefield and the transmitted wavefield are written as

$$\begin{aligned} \hat{E}_1^i &= \hat{e}_1^i \exp(-\gamma^{(1)} x_3) \, , \\ \hat{E}_1^r &= R_\perp \, \hat{e}_1^i \exp(\gamma^{(1)} x_3) \, , \\ \hat{E}_1^t &= T_\perp \, \hat{e}_1^i \exp(-\gamma^{(1)} x_3) \, , \end{aligned}$$

where the reflection and transmission coefficients are given by

$$R_\perp = \frac{\frac{Y^{(1)} - Y^{(2)}}{Y^{(1)} + Y^{(2)}} [1 - \exp(-2\gamma^{(2)} d)]}{1 - \left(\frac{Y^{(1)} - Y^{(2)}}{Y^{(1)} + Y^{(2)}}\right)^2 \exp(-2\gamma^{(2)} d)} \, , \quad T_\perp = \frac{\frac{4Y^{(1)} Y^{(2)}}{(Y^{(1)} + Y^{(2)})^2} \exp(\gamma^{(1)} d - \gamma^{(2)} d)}{1 - \left(\frac{Y^{(1)} - Y^{(2)}}{Y^{(1)} + Y^{(2)}}\right)^2 \exp(-2\gamma^{(2)} d)} \, .$$

For a single-frequency component ($s = j\omega$, $\omega = 2\pi f$), the shielding effectiveness is expressed as $S = 1/|T_\perp|$, or

$$S_{\mathrm{dB}} = -20 \, {}^{10}\!\log |T_\perp| \ \text{ decibel} \, .$$

The transmission line equivalent (→ Fig. 3.4):

Consider a length a in the x_1-direction and a length w in the x_2-direction of the one-dimensional wave. Introducing the electric potential $\hat{V}$ and the electric current $\hat{I}$, i.e.,

$$\hat{E}_1(x_3,s) = \frac{1}{a}\,\hat{V}(x_3,s)\,, \quad \hat{H}_2(x_3,s) = \frac{1}{w}\,\hat{I}(x_3,s)\,,$$

the one-dimensional Maxwell equations

$$\begin{aligned} \partial_3\hat{H}_2 + (\sigma{+}s\varepsilon)\hat{E}_1 &= \hat{I}_\Delta(s)\delta(x_3)\,,\\ \partial_3\hat{E}_1 + s\mu\hat{H}_2 &= 0\,, \end{aligned}$$

transfer into the one-dimensional transmission-line equations

$$\begin{aligned} \partial_3\hat{I} + (\mathcal{G}{+}s\mathcal{C})\hat{V} &= \hat{I}_\Delta(s)w\delta(x_3)\,,\\ \partial_3\hat{V} + s\mathcal{L}\,\hat{I} &= 0\,, \end{aligned}$$

where, per unit length of the transmission line, $\mathcal{G} = \sigma w/a$ is the *conductance*, $\mathcal{C} = \varepsilon w/a$ is the *capacitance* and $\mathcal{L} = \mu a/w$ is the *inductance*. Away from the source at $x_3 = 0$, along the transmission line, a wave propagates with electric potential and electric current

$$\begin{aligned} \hat{V} &= \tfrac{1}{2}Z_0\hat{I}_\Delta w\exp(-\gamma x_3) \quad \text{for } x_3 > 0\,,\\ \hat{I} &= \tfrac{1}{2}\hat{I}_\Delta w\exp(-\gamma x_3) \quad \text{for } x_3 > 0\,, \end{aligned}$$

where

$$\gamma = [(\mathcal{G}+s\mathcal{C})s\mathcal{L}]^{\frac{1}{2}} = [(\sigma+s\varepsilon)s\mu]^{\frac{1}{2}}\,, \quad \text{with } \mathrm{Re}(\gamma) \geq 0\,,$$

is the *propagation coefficient*, and

$$Z_0 = \left(\frac{s\mathcal{L}}{\mathcal{G}+s\mathcal{C}}\right)^{\frac{1}{2}} = \left(\frac{s\mu}{\sigma+s\varepsilon}\right)^{\frac{1}{2}}\frac{a}{w}$$

is the *characteristic impedance* of the transmission line.

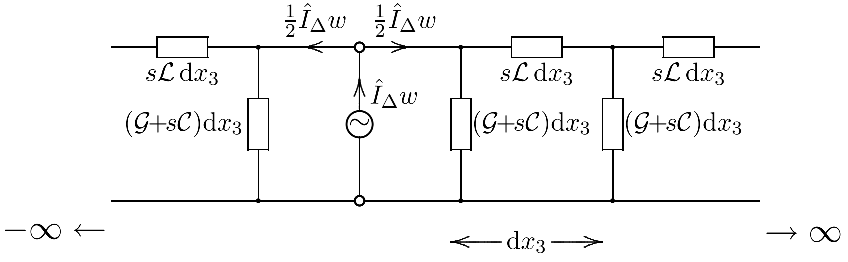


Figure 3.4. Distributed circuit representation of the transmission line.

For a *parallel-plate waveguide* with width a in the x_1-direction and characteristic length w in the x_2-direction, the *characteristic impedance* is found as

$$Z_0 = \left(\frac{s\mu}{\sigma+s\varepsilon}\right)^{\frac{1}{2}}\frac{a}{w}\,.$$

For a *coaxial line* with an inner conductor of radius a and an outer conductor of radius b, the *characteristic impedance* is found as

$$Z_0 = \left(\frac{s\mu}{\sigma+s\varepsilon}\right)^{\frac{1}{2}}\frac{\ln(b/a)}{2\pi}\,.$$

Propagation properties (→ Fig. 6.2):

The electric potential and current along the transmission line is given as a superposition of a wave propagating in the positive x_3-direction and a wave propagating in the negative x_3-direction, viz.,

$$\begin{aligned} \hat{V}(x_3,s) &= \hat{v}^+(s)[\exp(-\gamma x_3) + \Gamma_L\exp(\gamma x_3 - 2\gamma L)]\,,\\ \hat{I}(x_3,s) &= Y_0\hat{v}^+(s)[\exp(-\gamma x_3) - \Gamma_L\exp(\gamma x_3 - 2\gamma L)]\,, \end{aligned}$$

where

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

is the reflection coefficient of the wave reflected at the load location $x_3 = L$. The *input impedance* Z_{in} of the transmission line is

$$Z_{in} = \frac{\hat{V}(0,s)}{\hat{I}(0,s)} = Z_0\frac{Z_L + Z_0\tanh(\gamma L)}{Z_0 + Z_L\tanh(\gamma L)}\,.$$

When $Z_L = Z_0$, the input impedance is equal to the characteristic impedance ($Z_{in} = Z_0$) and there is no reflected wave propagating in the negative x_3-direction.

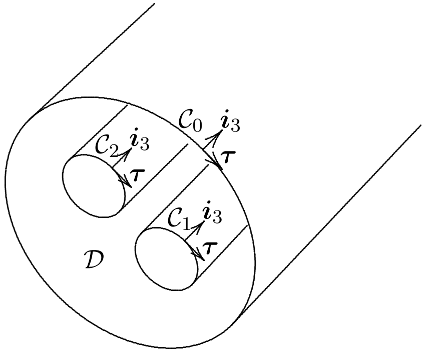


Figure 6.1. The multiconductor transmission line.

in which the real transversal functions $e_1 = e_1(x_1, x_2)$, $e_2 = e_2(x_1, x_2)$, $h_1 = h_1(x_1, x_2)$ and $h_2 = h_2(x_1, x_2)$ are normalized as

$$\iint_{(x_1,x_2)\in\mathcal{D}} (\boldsymbol{e} \times \boldsymbol{h}) \cdot \boldsymbol{i}_3 \,\mathrm{dA} = \iint_{(x_1,x_2)\in\mathcal{D}} (e_1 h_2 - e_2 h_1) \,\mathrm{dA} = 1 \,.$$

The electric-potential function $\hat{V} = \hat{V}(x_3, s)$ and the electric-current function $\hat{I} = \hat{I}(x_3, s)$ satisfy

$$\begin{aligned} \partial_3 \hat{I} + \gamma Y_0 \hat{V} &= 0 \,, \\ \partial_3 \hat{V} + \gamma Z_0 \hat{I} &= 0 \,, \end{aligned}$$

with $Z_0 = 1/Y_0$ and $\gamma = [(\sigma + s\varepsilon)s\mu]^{\frac{1}{2}}$, with $\mathrm{Re}(\gamma) \geq 0$.

The transversal functions are related to each other as

$$h_1 = -\left(\frac{\sigma + s\varepsilon}{s\mu}\right)^{\frac{1}{2}} Z_0 e_2 \,, \quad h_2 = \left(\frac{\sigma + s\varepsilon}{s\mu}\right)^{\frac{1}{2}} Z_0 e_1 \,,$$

and may be written as

$$e_1 = -\partial_1 \varphi \,, \quad e_2 = -\partial_2 \varphi \,.$$

The function $\varphi = \varphi(x_1, x_2)$ satisfies the two-dimensional Laplace equation

$$\partial_1 \partial_1 \varphi + \partial_2 \partial_2 \varphi = 0 \ \text{ in } \mathcal{D} \,,$$

supplemented with boundary conditions $\varphi =$ constant on the conductors.

4. Two-dimensional Electromagnetic Waves

Plane waves:

A plane wave propagating in the positive x_1- and x_3-direction is written as

$$\left.\begin{aligned} \hat{\boldsymbol{E}} &= \hat{\boldsymbol{e}}(s) \exp(-\gamma_1 x_1 - \gamma_3 x_3) \,, \\ \hat{\boldsymbol{H}} &= \hat{\boldsymbol{h}}(s) \exp(-\gamma_1 x_1 - \gamma_3 x_3) \,, \end{aligned}\right| \quad \boldsymbol{\gamma} \cdot \boldsymbol{\gamma} = \gamma_1^2 + \gamma_3^2 = (\sigma + s\varepsilon)s\mu \,.$$

In case of *steady-state*, the complex propagation vector $\boldsymbol{\gamma}$ is written as

$$\boldsymbol{\gamma}(j\omega) = \boldsymbol{\alpha}(\omega) + j\boldsymbol{\beta}(\omega) \,,$$

where $\boldsymbol{\alpha} = \{\alpha_1, 0, \alpha_3\}$ is the *attenuation vector* and $\boldsymbol{\beta} = \{\beta_1, 0, \beta_3\}$ is the *phase vector*. For *uniform* plane waves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have the same direction.

Uniform plane waves:

Let $\boldsymbol{s} = s_1 \boldsymbol{i}_1 + s_3 \boldsymbol{i}_3$ be a unit vector, then a uniform plane wave propagating in the $\boldsymbol{s}$-direction is written as

$$\left.\begin{aligned} \hat{\boldsymbol{E}} &= \hat{\boldsymbol{e}}(s) \exp[-\gamma(s_1 x_1 + s_3 x_3)] \,, \\ \hat{\boldsymbol{H}} &= \hat{\boldsymbol{h}}(s) \exp[-\gamma(s_1 x_1 + s_3 x_3)] \,, \end{aligned}\right| \quad \gamma^2 = (\sigma + s\varepsilon)s\mu \,.$$

The electric-field vector $\hat{\boldsymbol{e}}$, the magnetic-field vector $\hat{\boldsymbol{h}}$ and the propagation-direction vector $\boldsymbol{s}$ form a mutually perpendicular and right-hand triad.

In case of *steady-state*, the complex propagation coefficient is obtained as

$$\gamma = \alpha + j\beta = [(\sigma + j\omega\varepsilon)j\omega\mu]^{\frac{1}{2}}, \ \mathrm{Re}[\ \ ]^{\frac{1}{2}} \geq 0 \,,$$

where α is the attenuation coefficient and β is the phase coefficient, while the wavelength follows from $\lambda = 2\pi/\beta$. The time average Poynting's vector is given by

$$\langle \boldsymbol{S} \rangle_T = \tfrac{1}{2}\mathrm{Re}\left[\hat{\boldsymbol{E}} \times \hat{\boldsymbol{H}}^*\right] = \boldsymbol{S}_0 \exp[-2\alpha(s_1 x_1 + s_3 x_3)] \,.$$

Parallelly polarized waves: $\hat{e}_1 \neq 0$, $\hat{e}_3 \neq 0$ and $\hat{h}_2 \neq 0$ (→ Fig. 8.2).
The electric field strengths follow from the fundamental unknown $\hat{h}_2$ as

$$\hat{e}_1 = \frac{\gamma_3}{\sigma + s\varepsilon}\,\hat{h}_2 \,, \quad \hat{e}_3 = \frac{-\gamma_1}{\sigma + s\varepsilon}\,\hat{h}_2 \,,$$

while for steady-state uniform planes the energy transfer follows from

$$\boldsymbol{S}_0 = \tfrac{1}{2}\mathrm{Re}\left[\hat{\boldsymbol{e}} \times \hat{\boldsymbol{h}}^*\right] = \tfrac{1}{2}\mathrm{Re}\left[Z(j\omega)\right] \hat{h}_2 \hat{h}_2^* \boldsymbol{s} \,.$$

Perpendicularly polarized waves: $\hat{h}_1 \neq 0$, $\hat{h}_3 \neq 0$ and $\hat{e}_2 \neq 0$ (→ Fig. 8.4). The magnetic field strengths follow from the fundamental unknown $\hat{e}_2$ as

$$\hat{h}_1 = \frac{-\gamma_3}{s\mu}\,\hat{e}_2 \,, \quad \hat{h}_3 = \frac{\gamma_1}{s\mu}\,\hat{e}_2 \,,$$

while for steady-state uniform planes the energy transfer follows from

$$\boldsymbol{S}_0 = \tfrac{1}{2}\mathrm{Re}\left[\hat{\boldsymbol{e}} \times \hat{\boldsymbol{h}}^*\right] = \tfrac{1}{2}\mathrm{Re}\left[Y(j\omega)\right] \hat{e}_2 \hat{e}_2^* \boldsymbol{s} \,.$$

Reflection by and transmission through a plane interface (→ Fig. 4.1):

The incident (plane) wave propagates in medium (1) from the emitter in the positive x_1- and x_3-direction; it is given by

$$\begin{aligned} \hat{\boldsymbol{E}}^i &= \hat{\boldsymbol{e}}^i \exp(-\gamma_1^i x_1 - \gamma_3^i x_3) \,, \\ \hat{\boldsymbol{H}}^i &= \hat{\boldsymbol{h}}^i \exp(-\gamma_1^i x_1 - \gamma_3^i x_3) \,. \end{aligned}$$

The reflected (plane) wave propagates in medium (1) from the interface in positive x_1-direction and negative x_3-direction; it is given by

$$\begin{aligned} \hat{\boldsymbol{E}}^r &= \hat{\boldsymbol{e}}^r \exp(-\gamma_1^i x_1 + \gamma_3^i x_3) \,, \\ \hat{\boldsymbol{H}}^r &= \hat{\boldsymbol{h}}^r \exp(-\gamma_1^i x_1 + \gamma_3^i x_3) \,. \end{aligned}$$

The transmitted (plane) wave propagates in medium (2) from the interface in positive x_1- and x_3-direction; it is given by

$$\begin{aligned} \hat{\boldsymbol{E}}^t &= \hat{\boldsymbol{e}}^t \exp(-\gamma_1^i x_1 - \gamma_3^t x_3) \,, \\ \hat{\boldsymbol{H}}^t &= \hat{\boldsymbol{h}}^t \exp(-\gamma_1^i x_1 - \gamma_3^t x_3) \,, \end{aligned}$$

For given γ_1^i, in the two media, the components of the propagation vectors perpendicular to the interface follows from (with $\mathrm{Re}[\ ]^{\frac{1}{2}} \geq 0$)

$$\gamma_3^i = \left[(\sigma^{(1)} + s\varepsilon^{(1)})s\mu^{(1)} - (\gamma_1^i)^2\right]^{\frac{1}{2}} \,, \quad \gamma_3^t = \left[(\sigma^{(2)} + s\varepsilon^{(2)})s\mu^{(2)} - (\gamma_1^i)^2\right]^{\frac{1}{2}} \,,$$

In a *radially layered medium* (→ Fig. 5.2) with $n = n(r)$, $r = (x_1^2 + x_3^2)^{\frac{1}{2}}$, the trajectory follows from Snell's law

$$r\, n(r) \sin(\theta) = r_0\, n(r_0) \sin(\theta_0) \,,$$

with the auxiliary condition

$$\partial_l [r\, n(r) \cos(\theta)] = \partial_r [r n(r)] \,.$$

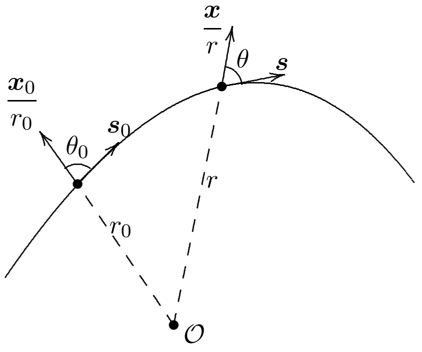


Figure 5.2. The ray trajectory of a uniform, electromagnetic ray in a radially layered medium.

6. Transmission Lines

TEM-waves:

A transverse electromagnetic (TEM) wave is a wave propagating in a source-free domain D in the longitudinal $\boldsymbol{i}_3$-direction (→ Fig. 6.1). The electric and magnetic field vectors at each point in space lie in a plane transverse to the direction of propagation. Hence,

$$\begin{aligned} \hat{E}_1 &= e_1(x_1,x_2)\,\hat{V}(x_3,s) \,, & \hat{H}_1 &= h_1(x_1,x_2)\,\hat{I}(x_3,s) \,, \\ \hat{E}_2 &= e_2(x_1,x_2)\,\hat{V}(x_3,s) \,, & \hat{H}_2 &= h_2(x_1,x_2)\,\hat{I}(x_3,s) \,, \\ \hat{E}_3 &= 0 \,, & \hat{H}_3 &= 0 \,, \end{aligned}$$

Uniform rays:
The unit vector $\boldsymbol{s}$ in the propagation direction of a (uniform) ray is a vector orthogonal to the surface $L =$ constant, i.e.,

$$s_1 = n^{-1}\partial_1 L_1 \,, \qquad s_3 = n^{-1}\partial_3 L_1 \,,$$

where $n = n(x_1, x_2) = c_0(\varepsilon\mu)^{\frac{1}{2}}$ is the index of refraction of the inhomogeneous medium.

Ray trajectories:
The ray curve $\boldsymbol{x} = \boldsymbol{x}(l) = x_1(l)\boldsymbol{i}_1 + x_3(l)\boldsymbol{i}_3$, where the parameter l is the arclength along the ray curve, satisfies the differential equation

$$\partial_l[n\partial_l x_1(l)] = \partial_1 n \,, \qquad \partial_l[n\partial_l x_3(l)] = \partial_3 n \,.$$

In a *horizontally layered medium* (→ Fig. 5.1) with $n = n(x_3)$, the trajectory follows from Snell's law

$$n(x_3)\sin(\theta) = n(x_{3;0})\sin(\theta_0) \,,$$

with the auxiliary condition

$$\partial_l[n\cos(\theta)] = \partial_3 n \,.$$

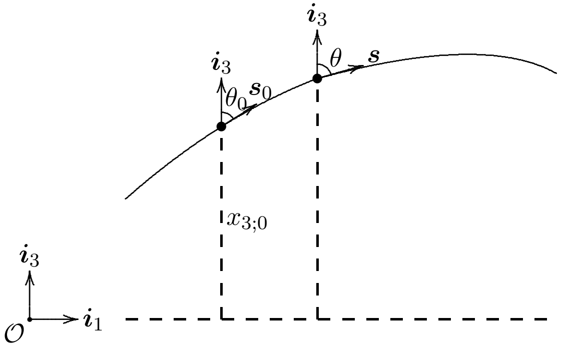


Figure 5.1. The ray trajectory of a uniform, electromagnetic ray in a horizontally layered medium.

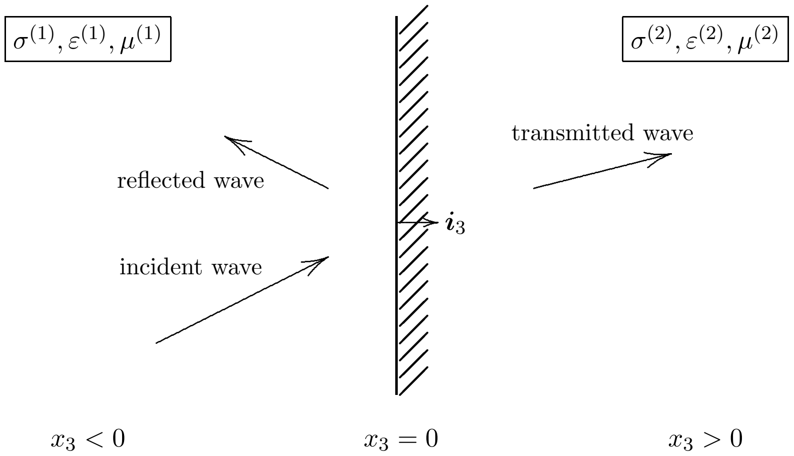


Figure 4.1. Reflection by and transmission through a plane interface.

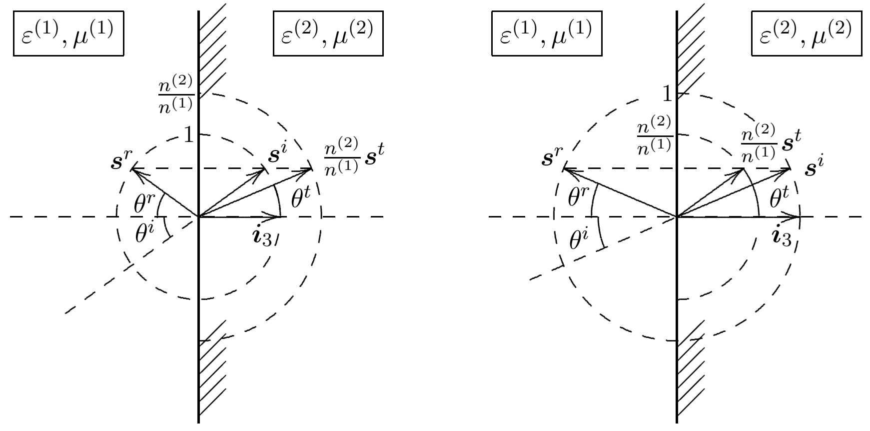
At the interface the electromagnetic field has to satisfy the boundary conditions that the tangential components of the electric and magnetic field strengths are continuous. From this the reflected and transmitted field strengths are expressed in terms of the incident field strengths through the reflection coefficients:

Parallel polarization: $\hat{h}_2^r = R_\| \, \hat{h}_2^i \,, \, \hat{h}_2^t = T_\| \, \hat{h}_2^i \,.$

$$R_\| = \frac{\dfrac{\gamma_3^i}{\sigma^{(1)} + s\varepsilon^{(1)}} - \dfrac{\gamma_3^t}{\sigma^{(2)} + s\varepsilon^{(2)}}}{\dfrac{\gamma_3^i}{\sigma^{(1)} + s\varepsilon^{(1)}} + \dfrac{\gamma_3^t}{\sigma^{(2)} + s\varepsilon^{(2)}}} \,, \quad T_\| = \frac{2\dfrac{\gamma_3^i}{\sigma^{(1)} + s\varepsilon^{(1)}}}{\dfrac{\gamma_3^i}{\sigma^{(1)} + s\varepsilon^{(1)}} + \dfrac{\gamma_3^t}{\sigma^{(2)} + s\varepsilon^{(2)}}} \,.$$

Perpendicular polarization: $\hat{e}_2^r = R_\perp \, \hat{e}_2^i \,, \, \hat{e}_2^t = T_\perp \, \hat{e}_2^i \,.$

$$R_\perp = \frac{\dfrac{\gamma_3^i}{\mu^{(1)}} - \dfrac{\gamma_3^t}{\mu^{(2)}}}{\dfrac{\gamma_3^i}{\mu^{(1)}} + \dfrac{\gamma_3^t}{\mu^{(2)}}} \,, \quad T_\perp = \frac{2\dfrac{\gamma_3^i}{\mu^{(1)}}}{\dfrac{\gamma_3^i}{\mu^{(1)}} + \dfrac{\gamma_3^t}{\mu^{(2)}}} \,.$$



(a) $n^{(1)} < n^{(2)}$ (b) $n^{(1)} > n^{(2)}$

Figure 4.2. Reflection and transmission of a uniform plane wave.

Steady-state uniform plane waves:

Introducing the angles of incidence and reflection (→ Fig. 4.2) we obtain *Snell's law of reflection*,

$$\theta^r = \theta^i \, .$$

Further, in the case of *lossless* media, we introduce the (real-valued) index of refraction $n^{(1)}$ of medium (1) and the index of refraction $n^{(2)}$ of medium (2) as

$$n^{(1)} = c_0[\varepsilon^{(1)}\mu^{(1)}]^{\frac{1}{2}} \, , \quad n^{(2)} = c_0[\varepsilon^{(2)}\mu^{(2)}]^{\frac{1}{2}} \, .$$

With the introduction of the angle of transmission (→ Fig. 4.2) we arrive at *Snell's law of refraction*,

$$n^{(1)}\sin(\theta^i) = n^{(2)}\sin(\theta^t) \, ,$$

provided that the angle of incidence θ^i is less than the critical angle θ^i_c, i.e.,

$$0 \le \sin(\theta^i) \le \sin(\theta^i_c) = \frac{n^{(2)}}{n^{(1)}} \qquad \text{for } n^{(1)} > n^{(2)} \, .$$

If $0 \le \theta^i \le \theta^i_c$, the transmitted wave is uniform and the expressions for the *Fresnel reflection and transmission coefficients* become

$$R_{\|} = \frac{\left(\frac{\mu^{(1)}}{\varepsilon^{(1)}}\right)^{\frac{1}{2}}\cos(\theta^i) - \left(\frac{\mu^{(2)}}{\varepsilon^{(2)}}\right)^{\frac{1}{2}}\cos(\theta^t)}{\left(\frac{\mu^{(1)}}{\varepsilon^{(1)}}\right)^{\frac{1}{2}}\cos(\theta^i) + \left(\frac{\mu^{(2)}}{\varepsilon^{(2)}}\right)^{\frac{1}{2}}\cos(\theta^t)} \, , \quad T_{\|} = \frac{2\left(\frac{\mu^{(1)}}{\varepsilon^{(1)}}\right)^{\frac{1}{2}}\cos(\theta^i)}{\left(\frac{\mu^{(1)}}{\varepsilon^{(1)}}\right)^{\frac{1}{2}}\cos(\theta^i) + \left(\frac{\mu^{(2)}}{\varepsilon^{(2)}}\right)^{\frac{1}{2}}\cos(\theta^t)} \, ,$$

$$R_{\perp} = \frac{\left(\frac{\varepsilon^{(1)}}{\mu^{(1)}}\right)^{\frac{1}{2}}\cos(\theta^i) - \left(\frac{\varepsilon^{(2)}}{\mu^{(2)}}\right)^{\frac{1}{2}}\cos(\theta^t)}{\left(\frac{\varepsilon^{(1)}}{\mu^{(1)}}\right)^{\frac{1}{2}}\cos(\theta^i) + \left(\frac{\varepsilon^{(2)}}{\mu^{(2)}}\right)^{\frac{1}{2}}\cos(\theta^t)} \, , \quad T_{\perp} = \frac{2\left(\frac{\varepsilon^{(1)}}{\mu^{(1)}}\right)^{\frac{1}{2}}\cos(\theta^i)}{\left(\frac{\varepsilon^{(1)}}{\mu^{(1)}}\right)^{\frac{1}{2}}\cos(\theta^i) + \left(\frac{\varepsilon^{(2)}}{\mu^{(2)}}\right)^{\frac{1}{2}}\cos(\theta^t)} \, .$$

It is possible that a Fresnel reflection coefficient vanishes for a particular value of the angle of incidence. The pertaining angle of incidence is called the *Brewster angle*. In the case of parallel polarization, and for dielectric media, where $\mu^{(1)} = \mu^{(2)} = \mu_0$, this Brewster angle follows from

$$\tan(\theta^i_B) = \left(\frac{\varepsilon^{(2)}}{\varepsilon^{(1)}}\right)^{\frac{1}{2}} \, , \quad \text{(for parallel polarization)} \, .$$

If $\theta^i > \theta^i_c$, the transmitted wave is non-uniform and we obtain *total reflection*, i.e., $|R_{\|}| = 1$ and $|R_{\perp}| = 1$.

5. Electromagnetic Rays in a Two-dimensional Medium

In a medium that is weakly *inhomogeneous* in the (x_1, x_3)-plane, the concept of electromagnetic rays is useful. We assume that the medium is invariant in the x_2-direction. Then, a twodimensional electromagnetic wavefield exists that is written as

$$\begin{aligned} \hat{\boldsymbol{E}}(x_1, x_3, s) &= \hat{\boldsymbol{e}}(x_1, x_3, s)\exp[-\frac{s}{c_0}L(x_1, x_3)] \, , \\ \hat{\boldsymbol{H}}(x_1, x_3, s) &= \hat{\boldsymbol{h}}(x_1, x_3, s)\exp[-\frac{s}{c_0}L(x_1, x_3)] \, , \end{aligned}$$

in which the *eikonal* $L = L(x_1, x_3)$ satisfies the *eikonal equation*:

$$(\partial_1 L)^2 + (\partial_3 L)^2 = c_0^2\varepsilon\mu \, .$$